COMPUTATIONAL DETAILS FOR THE PARAMETRIZATION OF $\mathrm{Irr}(UF_4(q))$ WITH $q = 2^f$

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Contents

We collect here the details for the computations on the parametrization of the sets $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ arising from nonabelian $[z, m, c]$-cores as in the paper *On characters of Sylow p-subgroups of finite Chevalley groups $G(p^f)$ over arbitrary primes*. The notation used here is the same as in Section 6 of the work. We include at the end of the file the Table 1.1 containing all labels for such nonabelian cores.

We repeatedly use in the sequel the following equation,

$$\phi(\sum_{h=1}^{\ell}\sum_{k=1}^{m} c_{h,k} s_{j_h}^{a_h} t_{i_k}^{b_k}) = 1, \tag{0.1}$$

which corresponds to Equation (5.1) in the above mentioned paper.

1. The nonabelian cores of $\mathrm{UF}_4(q)$, with $q = 2^f$

1.1. The $[2, 4, 1]$-core $\mathcal{F}_1$. We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_6, \alpha_9\}$,
- $\mathcal{Z} = \{\alpha_6, \alpha_9\}$,
- $\mathcal{A} = \mathcal{L} = \emptyset$,

- $\mathcal{I} = \{\alpha_2\}$ and $\mathcal{J} = \{\alpha_3\}$. The form of Equation (0.1) is

$$\phi(a_9 s_3 t_2^2 + a_6 s_3 t_2) = 1.$$

  We have that

$$X' = \{1, x_2(a_6/a_9)\}, \qquad Y' = \{1, x_3(a_9/a_6^2)\}.$$

  Hence we get the family

$$\mathcal{F}_1 = \{\chi_{c_2,c_3}^{a_6,a_9} \mid a_6, a_9 \in \mathbb{F}_q^\times, c_2, c_3 \in \mathbb{F}_2\}$$

  of $4(q-1)^2$ irreducible characters of degree $q/2$.

1.2. **The** $[3, 10, 9]$**-core** $\mathcal{F}_2$**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{12}, \alpha_{13}\}$,
- $\mathcal{Z} = \{\alpha_8, \alpha_{12}, \alpha_{13}\}$,
- $\mathcal{A} = \{\alpha_4\}$ and $\mathcal{L} = \{\alpha_9\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_3, \alpha_7\}$ and $\mathcal{J} = \{\alpha_5, \alpha_6, \alpha_{10}\}$.

This core is isomorphic to the $[3, 10, 9]$-core in type D_4. Hence we get the family

$$\mathcal{F}_2^1 = \{\chi^{a_8,a_{12},a_{13}} \mid a_8, a_{12}, a_{13} \in \mathbb{F}_q^\times\}$$

of $(q-1)^3$ irreducible characters of degree q^3, and

$$\mathcal{F}_2^2 = \{\chi_{c_{1,3,7},c_2}^{a_{5,6,10},a_8,a_{12},a_{13}} \mid a_{5,6,10}, a_8, a_{12}, a_{13} \in \mathbb{F}_q^\times, c_{1,3,7}, c_2 \in \mathbb{F}_2\}$$

of $4(q-1)^4$ irreducible characters of degree $q^3/2$.

1.3. **The** $[4, 8, 2]$**-core** $\mathcal{F}_3$**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_{12}, \alpha_{18}\}$,
- $\mathcal{Z} = \{\alpha_6, \alpha_9, \alpha_{12}, \alpha_{18}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{11}, \alpha_{16}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_5\}$ and $\mathcal{J} = \{\alpha_3, \alpha_7\}$.

Notice that

$$X_{\mathcal{S}} = X_{\mathcal{S}_1} \times X_{\mathcal{S}_2}, \qquad \text{where} \qquad \mathcal{S}_1 = \{\alpha_2, \alpha_3, \alpha_6, \alpha_9\}, \mathcal{S}_2 = \{\alpha_5, \alpha_7, \alpha_{12}, \alpha_{18}\},$$

and $X_{\mathcal{S}_1}$, $X_{\mathcal{S}_2}$ have the configuration of a $[2, 4, 1]$-core. Hence we get the family

$$\mathcal{F}_3 = \{\chi_{c_2,c_3,c_5,c_7}^{a_6,a_9,a_{12},a_{18}} \mid a_6, a_9, a_{12}, a_{18} \in \mathbb{F}_q^\times, c_2, c_3, c_5, c_7 \in \mathbb{F}_2\}$$

of $16(q-1)^4$ irreducible characters of degree $q^2/4$.

1.4. **The** $[4, 8, 4]$**-core** $\mathcal{F}_{4,1}$**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{18}\}$,
- $\mathcal{Z} = \{\alpha_6, \alpha_8, \alpha_{10}, \alpha_{18}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{11}, \alpha_{16}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_5\}$ and $\mathcal{J} = \{\alpha_3, \alpha_7\}$.

The form of Equation (0.1) is

$$\phi(s_3(a_6t_2 + a_8t_5) + s_7(a_{18}t_5s_7 + a_{10}t_2)) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_6t_2 = a_8t_5 \\ a_{10}^2t_2^2 = a_{18}t_5 \end{cases}, \qquad \text{resp.} \qquad \begin{cases} a_6s_3 = a_{10}s_7 \\ a_{18}s_7^2 = a_8s_3 \end{cases}.$$

We have that

$$X' = \{1, x_2(\frac{a_6a_{18}}{a_8a_{10}^2})x_5(\frac{a_6^2a_{18}}{a_8^2a_{10}^2})\} \qquad \text{and} \qquad Y' = \{1, x_3(\frac{a_8a_{10}^2}{a_6^2a_{18}})x_7(\frac{a_8a_{10}}{a_6a_{18}})\}.$$

This gives the family

$$\mathcal{F}_{4,1} = \{\chi_{c_{2,5},c_{3,7}}^{a_6,a_8,a_{10},a_{18}} \mid a_6, a_8, a_{10}, a_{18} \in \mathbb{F}_q^\times, c_{2,5}, c_{3,7} \in \mathbb{F}_2\}$$

of $4(q-1)^4$ irreducible characters of degree $q^2/2$.

1.5. **The** $[4, 8, 4]$**-core** $\mathcal{F}_{4,2}$**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{18}\}$,
- $\mathcal{Z} = \{\alpha_8, \alpha_9, \alpha_{10}, \alpha_{18}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{11}, \alpha_{16}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_5\}$ and $\mathcal{J} = \{\alpha_3, \alpha_7\}$.

The form of Equation (0.1) is

$$\phi(s_3(a_9t_2s_3 + a_8t_5) + s_7(a_{18}t_5s_7 + a_{10}t_2)) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_8^2t_5^2 = a_9t_2 \\ a_{10}^2t_2^2 = a_{18}t_5 \end{cases} \Leftrightarrow \begin{cases} t_2 = a_8^2a_9^{-1}t_5^2 \\ t_5^4 = a_8^{-4}a_9^2a_{10}^{-2}a_{18}t_5 \end{cases}, \text{resp.} \begin{cases} a_9s_3^2 = a_{10}s_7 \\ a_{18}s_7^2 = a_8s_3 \end{cases} \Leftrightarrow \begin{cases} s_7 = a_9a_{10}^{-1}s_3^2 \\ s_3^4 = a_8a_9^{-2}a_{10}^2a_{18}^{-1}s_3 \end{cases}.$$

Let us assume that $f = 2k$. If $a_{18} \notin a_8a_9^{-2}a_{10}^2\mathbb{F}_{q,3}^\times$, then the quartic equations above just have a trivial solution. Hence we have in this case $X' = 1$, $Y' = 1$, and we get the family

$$\mathcal{F}_{4,2}^{f \text{ even},1} = \{\chi^{a_8,a_9,a_{10},a_{18}^1} \mid a_8, a_9, a_{10} \in \mathbb{F}_q^\times, a_{18}^1 \notin a_8a_9^{-2}a_{10}^2\mathbb{F}_{q,3}^\times\}$$

of $2(q-1)^4/3$ irreducible characters of degree q^2. If $a_{18} \in a_8a_9^{-2}a_{10}^2\mathbb{F}_{q,3}^\times$, then there are three distinct values $\omega_{8,9,10,18;i}$, $i = 1, 2, 3$, such that $\omega_{8,9,10,18;i}^3 = a_8a_9^{-2}a_{10}^2a_{18}^{-1}$. In this case, we have

$$X' = \{1\} \cup \{x_2(a_8a_9^{-1}\omega_{8,9,10,18;i}^{-2})x_5(a_8^{-1}\omega_{8,9,10,18;i}^{-1}) \mid i \in [1, 3]\}\},$$

and

$$Y' = \{1\} \cup \{x_3(\omega_{8,9,10,18;i})x_7(a_9a_{10}^{-1}\omega_{8,9,10,18;i}^2) \mid i \in [1, 3]\}.$$

We now observe that X' and Y' are cyclic of order 4. We get the family

$$\mathcal{F}_{4,2}^{f \text{ even},2} = \{\chi_{r_{2,5},r_{3,7}}^{a_8,a_9,a_{10},a_{18}^2} \mid a_8, a_9, a_{10} \in \mathbb{F}_q^\times, a_{18}^2 \in a_8a_9^{-2}a_{10}^2\mathbb{F}_{q,3}^\times, r_{2,5}, r_{3,7} \in \mathbb{Z}_4\}$$

of $16(q-1)^4/3$ irreducible characters of degree $q^2/4$.

Let us assume that $f = 2k+1$. Let $\omega_{8,9,10,18}$ be the unique cube root of $a_8a_9^{-2}a_{10}^2a_{18}^{-1}$. Then we get

$$X' = \{1, x_2(a_8a_9^{-1}\omega_{8,9,10,18}^{-2})x_5(a_8^{-1}\omega_{8,9,10,18}^{-1})\}, \qquad Y' = \{1, x_3(\omega_{8,9,10,18})x_7(a_9a_{10}^{-1}\omega_{8,9,10,18}^2)\}.$$

Hence we get the family

$$\mathcal{F}_{4,2}^{f\ \text{odd}} = \{\chi_{c_{2,5},c_{3,7}}^{a_8,a_9,a_{10},a_{18}} \mid a_8, a_9, a_{10}, a_{18} \in \mathbb{F}_q^\times, c_{2,5}, c_{3,7} \in \mathbb{F}_2\}$$

of $4(q-1)^4$ irreducible characters of degree $q^2/2$.

1.6. **The** $[4, 10, 5]$**-core** $\mathcal{F}_5$**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{13}, \alpha_{16}\}$,
- $\mathcal{Z} = \{\alpha_5, \alpha_8, \alpha_{13}, \alpha_{16}\}$,
- $\mathcal{A} = \{\alpha_3\}$ and $\mathcal{L} = \{\alpha_{10}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_4, \alpha_7\}$ and $\mathcal{J} = \{\alpha_2, \alpha_6, \alpha_9\}$.

The form of Equation (0.1) is

$$\phi(s_2(a_{16}t_7^2 + a_5t_1) + s_6(a_8t_1 + a_{13}t_7) + s_9(a_{16}t_4^2 + a_{13}t_4)) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_{16}t_7^2 = a_5t_1 \\ a_8t_1 = a_{13}t_7 \\ a_{16}t_4^2 = a_{13}t_4 \end{cases} \Leftrightarrow \begin{cases} t_1 = a_5^{-1}a_{16}t_7^2 \\ t_7^2 = a_5a_8^{-1}a_{13}a_{16}^{-1}t_7 \\ a_{16}t_4^2 = a_{13}t_4 \end{cases}, \text{resp.} \begin{cases} a_5s_2 = a_8s_6 \\ s_9^2a_{13}^2 = s_9a_{16} \\ s_2^2a_5^2 = s_2a_{16} \end{cases} \Leftrightarrow \begin{cases} s_7 = a_9a_{10}^{-1}s_3^2 \\ s_3^4 = a_8a_9^{-2}a_{10}^2a_{18}^{-1}s_3 \end{cases}.$$

Hence we get

$$X' = \{1, x_4(a_{13}a_{16}^{-1})\} \cup \{1, x_1(a_5a_8^{-2}a_{13}^2a_{16}^{-1})x_7(a_5a_8^{-1}a_{13}a_{16}^{-1})\},$$

and

$$Y' = \{1, x_9(a_{13}^{-2}a_{16})\} \cup \{1, x_2(a_5^{-1}a_8^{-1}a_{16})x_6(a_5^{-2}a_{16})\}.$$

Observe now that X' and Y' are both isomorphic to $C_2 \times C_2$. Hence we get the family

$$\mathcal{F}_5 = \{\chi_{c_{1,7},c_{2,6},c_4,c_9}^{a_5,a_8,a_{13},a_{16}} \mid a_5, a_8, a_{13}, a_{16} \in \mathbb{F}_q^\times, c_{1,7}, c_{2,6}, c_4, c_9 \in \mathbb{F}_2\}$$

of $16(q-1)^4$ irreducible characters of degree $q^3/4$.

1.7. **The** $[4, 11, 6]$**-core** $\mathcal{F}_6$**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_4, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{15}, \alpha_{16}, \alpha_{19}, \alpha_{24}\}$,
- $\mathcal{Z} = \{\alpha_{10}, \alpha_{16}, \alpha_{19}, \alpha_{24}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_8, \alpha_{11}, \alpha_{14}\}$ and $\mathcal{L} = \{\alpha_{13}, \alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{22}, \alpha_{23}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_6, \alpha_9\}$ and $\mathcal{J} = \{\alpha_4, \alpha_7, \alpha_{12}, \alpha_{15}\}$.

The form of Equation (0.1) is

$$\phi(s_4(a_{16}t_9s_4 + a_{10}t_6) + s_7(a_{16}t_2s_7 + a_{10}t_2) + s_{12}(a_{24}t_9s_{12} + a_{19}t_9) + s_{15}(a_{24}t_2s_{15} + a_{19}t_6)) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_{10}^2t_6^2 = a_{16}t_9 \\ a_{10}^2t_2^2 = a_{16}t_2 \\ a_{19}^2t_9^2 = a_{24}t_9 \\ a_{19}^2t_6^2 = a_{24}t_2 \end{cases}, \text{resp.} \begin{cases} a_{16}s_7^2 + a_{24}s_{15}^2 = a_{10}s_7 \\ a_{10}s_4 = a_{19}s_{15} \\ a_{16}s_4^2 + a_{24}s_{12}^2 = a_{19}s_{12} \end{cases}.$$

Exclude the trivial solution $(0,0,0)$ in the t_i and s_j. Then the systems can be rewritten in the following way,

$$\begin{cases} t_2 = a_{10}^{-2}a_{16} \\ t_9 = a_{19}^{-2}a_{24} \\ t_6^2 = a_{10}^{-2}a_{16}a_{19}^{-2}a_{24} \\ t_6^2 = a_{10}^{-2}a_{16}a_{19}^{-2}a_{24} \end{cases}, \text{resp.} \begin{cases} s_{15} = a_{10}a_{19}^{-1}s_4 \\ s_4^2 = a_{16}^{-1}a_{24}s_{12}^2 + a_{16}^{-1}a_{19}s_{12} \\ (a_{16}a_{19}s_7 + a_{10}a_{24}s_{12})^2 = a_{10}a_{19}(a_{16}a_{19}s_7 + a_{10}a_{24}s_{12}) \end{cases}.$$

Hence we have that

$$X' = \{1, x_2(a_{10}^{-2}a_{16})x_6(a_{10}\omega_{16}a_{19}\omega_{24})x_9(a_{19}^{-2}a_{24})\},$$

and

$$Y' = \{1, x_4(s)x_7(a_{10}a_{16}^{-1}a_{19}^{-1}a_{24}t^2+c)x_{12}(t^2)x_{15}(a_{10}a_{19}^{-1}s) \mid t \in \mathbb{F}_q, s = \omega_{16}^{-1}\omega_{24}t^2+\omega_{16}^{-1}\omega_{19}t, c \in \{0, a_{10}^{-1}a_{16}\}\}.$$

Hence we get the family

$$\mathcal{F}_6 = \{\chi_{b_{4,7,12,15},c_{2,6,9},c_{4,7,12,15}}^{a_{10},a_{16},a_{19},a_{24}} \mid a_{10}, a_{16}, a_{19}, a_{24} \in \mathbb{F}_q^\times, b_{4,7,12,15} \in \mathbb{F}_q, c_{2,6,9}, c_{4,7,12,15} \in \mathbb{F}_2\}$$

of $4q(q-1)^4$ irreducible characters of degree $q^3/2$.

1.8. **The** $[4,12,9]$**-core** $\mathcal{F}_{7,1}$**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{19}, \alpha_{23}\}$,
- $\mathcal{Z} = \{\alpha_{12}, \alpha_{15}, \alpha_{19}, \alpha_{23}\}$,
- $\mathcal{A} = \{\alpha_2, \alpha_3, \alpha_6, \alpha_9, \alpha_{14}\}$ and $\mathcal{L} = \{\alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_8, \alpha_{11}\}$ and $\mathcal{J} = \{\alpha_4, \alpha_7, \alpha_{10}, \alpha_{13}\}$.

The form of Equation (0.1) is

$$\phi(s_4(a_{12}t_8 + a_{15}t_{11}) + s_7(a_{12}t_5 + a_{15}t_8) + s_{10}(a_{23}t_{11}s_{10} + a_{12}t_1 + a_{19}t_{11}) + s_{13}(a_{23}t_5s_{13} + a_{15}t_1 + a_{19}t_8)) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_{12}t_8 = a_{15}t_{11} \\ a_{12}t_5 = a_{15}t_8 \\ a_{12}^2t_1^2 + a_{19}^2t_{11}^2 = a_{23}t_{11} \\ a_{15}^2t_1^2 + a_{19}^2t_8^2 = a_{23}t_5 \end{cases}, \text{resp.} \begin{cases} a_{12}s_{10} = a_{15}s_{13} \\ a_{12}s_7 = a_{23}s_{13}^2 \\ a_{12}s_4 + a_{15}s_7 = a_{19}s_{13} \\ a_{23}s_{10}^2 = a_{15}s_4 + a_{19}s_{10} \end{cases}.$$

The systems can be rewritten in the following way,

$$\begin{cases} t_{11} = a_{12}a_{15}^{-1}t_8 \\ t_5 = a_{12}^{-1}a_{15}t_8 \\ a_{12}^2a_{15}^2t_1^2 + a_{12}^2a_{19}^2t_8^2 = a_{12}a_{15}a_{23}t_8 \\ a_{12}a_{15}^2t_1^2 + a_{12}a_{19}^2t_8^2 = a_{15}a_{23}t_8 \end{cases}, \text{resp.} \begin{cases} s_{10} = a_{12}^{-1}a_{15}s_{13} \\ s_7 = a_{12}^{-1}a_{23}s_{13}^2 \\ a_{12}s_4 = a_{12}^{-1}a_{15}a_{23}s_{13}^2 + a_{19}s_{13} \\ a_{15}s_4 = a_{12}^{-1}a_{15}^2a_{23}s_{13}^2 + a_{12}^{-1}a_{15}a_{19}s_{13} \end{cases}.$$

Hence we have that

$$X' = \{x_1(a_{15}^{-1}a_{19}t^2 + \omega_{12}^{-1}\omega_{15}^{-1}\omega_{23}t)x_5(a_{12}^{-1}a_{15}t^2)x_8(t^2)x_{11}(a_{12}a_{15}^{-1}t^2) \mid t \in \mathbb{F}_q\},$$

and

$$Y' = \{x_4(a_{12}^{-2}a_{15}a_{23}t^2 + a_{12}^{-1}a_{19}t)x_7(a_{12}^{-1}a_{23}t^2)x_{10}(a_{12}^{-1}a_{15}t)x_{13}(t) \mid t \in \mathbb{F}_q\}.$$

Hence we get the family

$$\mathcal{F}_{7,1} = \{\chi_{b_{1,5,8,11},b_{4,7,10,13}}^{a_{12},a_{15},a_{19},a_{23}} \mid a_{12}, a_{15}, a_{19}, a_{23} \in \mathbb{F}_q^\times, b_{1,5,8,11}, b_{4,7,10,13} \in \mathbb{F}_q\}$$

of $q^2(q-1)^4$ irreducible characters of degree q^3.

1.9. **The** $[4,12,9]$**-core** $\mathcal{F}_{7,2}$**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{16}\}$,
- $\mathcal{Z} = \{\alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{16}\}$,
- $\mathcal{A} = \mathcal{L} = \varnothing$,
- $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\mathcal{J} = \{\alpha_5, \alpha_6, \alpha_7, \alpha_9\}$.

The form of Equation (0.1) is

$$\phi(s_5(a_{11}t_3^2 + a_8t_3) + s_6(a_8t_1 + a_{10}t_4) + s_7(a_{16}t_2s_7 + a_{10}t_2) + s_9(a_{16}t_4^2 + a_{11}t_1)) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_{11}t_3^2 = a_8t_3 \\ a_8t_1 = a_{10}t_4 \\ a_{10}^2t_2^2 = a_{16}t_2 \\ a_{16}t_4^2 = a_{11}t_1 \end{cases}, \text{resp.} \begin{cases} a_8s_6 = a_{11}s_9 \\ a_{16}s_7^2 = a_{10}s_7 \\ a_8^2s_5^2 = a_{11}s_5 \\ a_{10}^2s_6^2 = a_{16}s_9 \end{cases}.$$

Hence we have that

$$X' = \{x_1(c_1)x_2(c_2)x_3(c_3)x_4(\frac{a_8}{a_{10}}c_1) \mid c_1 \in \{0, \frac{a_{10}^2a_{11}}{a_8^2a_{16}}\}, c_2 \in \{0, \frac{a_{16}}{a_{10}^2}\}, c_3 \in \{0, \frac{a_8}{a_{11}}\}, \},$$

and

$$Y' = \{x_5(c_1)x_6(c_2)x_7(c_3)x_9(\frac{a_8}{a_{11}}c_2) \mid c_1 \in \{0, \frac{a_{11}}{a_8^2}\}, c_2 \in \{0, \frac{a_8a_{16}}{a_{10}^2a_{11}}\}, c_3 \in \{0, \frac{a_{10}}{a_{16}}\}, \}$$

with $X' = X_1'X_2'X_3'$ in a natural way with $X_1' \subseteq X_2$, $X_2' \subseteq X_3$, and $X_3' \subseteq X_1X_4$ and $Y' = Y_1'Y_2'Y_3'$ with $Y_1' \subseteq X_7$, $Y_2' \subseteq X_5$, and $Y_3' \subseteq X_6X_9$. Now notice that Y' is a group in direct product in the new group, and that X' is *not always* a group, namely

$$[x_2(s_2), x_3(t_3)] = x_6(s_2t_7)x_9(s_2t_3^2), \quad [x_1(s_1)x_4(s_4), x_2(t_2)] = x_5(s_1t_2)$$

and

$$[x_1(s_1)x_4(s_4), x_3(t_3)] = x_7(s_4t_3),$$

that is,

$$[x_2(\frac{a_{16}}{a_{10}^2}), x_3(\frac{a_8}{a_{11}})] = x_6(\frac{a_8a_{16}}{a_{10}^2a_{11}})x_9(\frac{a_8^2a_{16}}{a_{10}^2a_{11}^2}), \quad [x_1(\frac{a_{10}^2a_{11}}{a_8^2a_{16}})x_4(\frac{a_{10}a_{11}}{a_8a_{16}}), x_2(\frac{a_{16}}{a_{10}^2})] = x_5(\frac{a_{11}}{a_8^2})$$

and

$$[x_1(\frac{a_{10}^2a_{11}}{a_8^2a_{16}})x_4(\frac{a_{10}a_{11}}{a_8a_{16}}), x_3(\frac{a_8}{a_{11}})] = x_7(\frac{a_{10}}{a_{16}}).$$

Hence we have that $[X_i', X_j'] = Y_k'$ for every i, j, k with $\{i, j, k\} = \{1, 2, 3\}$. Recall that if $Z = X_{\mathcal{Z}}/\ker\lambda$, and for $c_1, c_2, c_3 \in \mathbb{F}_2$ we call $\lambda^{\underline{c}} := \lambda^{c_1,c_2,c_3}$ the extension of λ to $X_{\mathcal{Z}}Y'$ such that $\lambda^{\underline{c}}(y_i) = y_i^c$ for every $y_i \in Y_i$ and $i = 1, 2, 3$, then inflation and induction over groups of order $q^4/8$ induces a bijection

$$\mathrm{Irr}(\mathfrak{C}_{7,2})_{\mathcal{Z}} \longrightarrow \bigsqcup_{c_1,c_2,c_3 \in \mathbb{F}_2} \mathrm{Irr}(X'Y'Z \mid \lambda^{\underline{c}}).$$

Let us assume $c_i = 1$ for every $i = 1, 2, 3$. Then we can apply the Reduction Lemma with arm X'_1 and leg X'_2. In this case, we get

$$\mathrm{Irr}(X'Y'Z \mid \lambda^{1,1,1}) \longrightarrow \mathrm{Irr}(X'_3Y'_1Y'_2Y'_3Z \mid \lambda^{1,1,1}),$$

and $X'_3Y'_1Y'_2Y'_3Z$ is abelian. Hence we get the family $\mathcal{F}^8_{7,2}$ of $2(q-1)^4$ irreducible characters of degree $q^4/4$ as in Table 1.1.

Let us now assume that $c_i = c_j = 1$ and $c_k = 0$ for any $\{i, j, k\} = \{1, 2, 3\}$. The Reduction Lemma now applies with arm X'_i and leg X'_k. We have

$$\mathrm{Irr}(X'Y'Z \mid \lambda^{\underline{c}}) \longrightarrow \mathrm{Irr}(X'_jY'_1Y'_2Y'_3Z \mid \lambda^{1,1,1}),$$

with $X'_jY'_1Y'_2Y'_3Z$ abelian. This gives the three families $\mathcal{F}^5_{7,2}$, $\mathcal{F}^6_{7,2}$ and $\mathcal{F}^7_{7,2}$, which yield in total $3\,(2(q-1)^4) = 6(q-1)^4$ irreducible characters of degree $q^4/4$.

Let us then assume that $c_i = 1$, and $c_j = c_k = 0$ for any $\{i, j, k\} = \{1, 2, 3\}$. The Reduction Lemma now applies with arm X'_j and leg X'_k. We have

$$\mathrm{Irr}(X'Y'Z \mid \lambda^{\underline{c}}) \longrightarrow \mathrm{Irr}(X'_iY'_1Y'_2Y'_3Z \mid \lambda^{1,1,1}),$$

with $X'_kY'_1Y'_2Y'_3Z$ abelian. This gives the three families $\mathcal{F}^2_{7,2}$, $\mathcal{F}^3_{7,2}$ and $\mathcal{F}^4_{7,2}$ as in Table 1.1; these yield in total $3\,(2(q-1)^4) = 6(q-1)^4$ irreducible characters of degree $q^4/4$.

Finally, let us assume $c_1 = c_2 = c_3 = 0$. Then we have that

$$\mathrm{Irr}(X'Y'Z \mid \lambda^{0,0,0}) \longrightarrow \mathrm{Irr}(X'Y'Z/Y' \mid \lambda),$$

and $X'Y'Z/Y' = X'_1X'_2X'_3ZY'/Y'$ is abelian. We have determined our family $\mathcal{F}^1_{7,2}$ of $8(q-1)^4$ irreducible characters of degree $q^4/8$ as in Table 1.1.

1.10. **The $[5, 9, 3]$-core $\mathcal{F}_8$.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{18}\}$,
- $\mathcal{Z} = \{\alpha_6, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{18}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{11}, \alpha_{16}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_7\}$ and $\mathcal{J} = \{\alpha_3, \alpha_5\}$.

The form of Equation (0.1) is

$$\phi(s_3(a_9t_2s_3 + a_6t_2) + s_5(a_{18}t_7^2 + a_{12}t_7) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_6^2t_2^2 = a_9t_2 \\ a_{18}t_7^2 = a_{12}t_7 \end{cases}, \text{resp.} \begin{cases} a_9s_3^2 = a_6s_3 \\ a_{12}^2s_5^2 = a_{18}s_5 \end{cases}.$$

Hence we have that

$$X' = \{x_2(c_1)x_7(c_2) \mid c_1 \in \{0, a_6^{-2}a_9\}, c_2 \in \{0, a_{18}^{-1}a_{12}\}\}$$

and

$$Y' = \{x_3(c_1)x_5(c_2) \mid c_1 \in \{0, a_6a_9^{-1}\}, c_2 \in \{0, a_{12}^{-2}a_{18}\}\},$$

with $X' = X'_1X'_2$ in a natural way with $X'_1 \subseteq X_2$ and $X'_2 \subseteq X_7$. Now notice that Y' is a group in direct product in the new group, and that X' is *not always* a group, namely

$$[x_2(s_2), x_7(t_7)] = x_{10}(s_2t_7).$$

Notice though that

$$\lambda([x_2(a_6^{-2}a_9), a_{18}^{-1}a_{12}]) = \phi\left(\frac{a_{10}a_9a_{12}}{a_6^2a_{18}}\right).$$

If $a_{10} \in \frac{a_6^2a_{18}}{a_9a_{12}} \ker\phi$, which happens for $(q-2)/2$ values of $a_{10} \in \mathbb{F}_q^\times$, then X' is an abelian group in the new subquotient. In this case, we get the family

$$\mathcal{F}_8^1 = \{\chi_{c_2,c_3,c_5,c_7}^{a_6,a_9,a_{10}^1,a_{12},a_{18}} \mid a_6, a_9, a_{12}, a_{18} \in \mathbb{F}_q^\times, a_{10}^1 \in \frac{a_6^2a_{18}}{a_9a_{12}} \ker\phi \setminus \{0\}, c_2, c_3, c_5, c_7 \in \mathbb{F}_2\}$$

of $8(q-2)(q-1)^4$ irreducible characters of degree $q^2/4$.

If $a_{10} \notin \frac{a_6^2a_{18}}{a_9a_{12}} \ker\phi$, which happens for $q/2$ values of $a_{10} \in \mathbb{F}_q^\times$, then we can apply the Reduction Lemma again with arm X_1' and leg X_2'. This gives the family

$$\mathcal{F}_8^2 = \{\chi_{c_3,c_5}^{a_6,a_9,a_{10}^2,a_{12},a_{18}} \mid a_6, a_9, a_{12}, a_{18} \in \mathbb{F}_q^\times, a_{10}^2 \in \mathbb{F}_q^\times \setminus \frac{a_6^2a_{18}}{a_9a_{12}} \ker\phi, c_3, c_5 \in \mathbb{F}_2\}$$

of $2q(q-1)^4$ irreducible characters of degree $q^2/2$.

1.11. **The** $[5,9,4]$**-core** $\mathcal{F}_{9,1}$**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{18}\}$,
- $\mathcal{Z} = \{\alpha_6, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{18}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{11}, \alpha_{16}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_5\}$ and $\mathcal{J} = \{\alpha_3, \alpha_7\}$.

The form of Equation (0.1) is

$$\phi(s_3(a_9t_2s_3 + a_6t_2 + a_8t_5) + s_7(a_{18}t_5s_7 + a_{10}t_2) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_6^2t_2^2 + a_8^2t_5^2 = a_9t_2 \\ a_{10}^2t_2^2 = a_{18}t_5 \end{cases}, \text{resp.} \begin{cases} a_9s_3^2 = a_6s_3 + a_{10}s_7 \\ a_{18}s_7^2 = a_8s_3 \end{cases}.$$

Exclude the trivial solution $(0,0)$ in the t_i and s_j. Then the systems can be rewritten in the following way,

$$\begin{cases} t_5 = a_{10}^2a_{18}^{-1}t_2^2 \\ t_2^3 + a_6^2a_8^{-2}a_{10}^{-4}a_{18}^2t_2 + a_8^{-2}a_9a_{10}^{-4}a_{18}^2 = 0 \end{cases}, \text{resp.} \begin{cases} s_3 = a_8^{-1}a_{18}s_7^2 \\ s_7^3 + a_6a_8a_9^{-1}a_{18}^{-1}s_7 + a_8^2a_9^{-1}a_{10}a_{18}^{-2} = 0 \end{cases}.$$

Let us fix a_8, a_{10}, a_{18}, and define a and b by

$$a_6^2 = \frac{a_8^2a_{10}^4}{a_{18}^2}a, \qquad a_9 = \frac{a_8^2a_{10}^4}{a_{18}^2}b.$$

For $i \in \{0,1,3\}$, we call

(1.1) $\qquad \mathcal{A}_i := \{(a_6, a_9) \in \mathbb{F}_q^\times \mid X^3 + a_6^2a_8^{-2}a_{10}^{-4}a_{18}^2X + a_8^{-2}a_9a_{10}^{-4}a_{18}^2 = 0 \text{ has } i \text{ solutions}\}$,

and fixed i and $(a_6^i, a_9^i) \in \mathcal{A}_i$ we denote by $\mathcal{T}_i$ the set of cardinality i of elements satisfying the equation 1.1. Let

(1.2) $\qquad \mathcal{A}_i := \{(a_6, a_9) \in \mathbb{F}_q^\times \mid X^3 + a_6a_8a_9^{-1}a_{18}^{-1}X + a_8^2a_9^{-1}a_{10}a_{18}^{-2} = 0 \text{ has } i \text{ solutions}\}$.

Now notice that if x_0 is $X^3+\omega_a X/b+1/b=0$, then bx_0^2 is a solution of $X^3+aX+b=0$, hence

$$\mathcal{A}_i=\{bx_0^2 \mid x_0\in\mathcal{A}_i'\},$$

in particular $|\mathcal{A}_i|=|\mathcal{A}_i'|$.

Fixed i and $(a_6^i,a_9^i)\in\mathcal{A}_i$ we denote by $\mathcal{S}_i$ the set of cardinality i of elements satisfying the equation 1.2. Let then $(a_6,a_9)=(\mathbb{F}_q^\times)^2$. Then $(a_6,a_9)=(a_6^i,a_9^i)$ for some $i\in\{0,1,3\}$; in this case, we get

$$X':=\{1\}\cup\{x_2(t)x_5(a_{10}^2a_{18}^{-1}t^2)\mid t\in\mathcal{T}_i\}$$

and

$$Y'=\{1\}\cup\{x_3(s)x_7(a_8^{-1}a_{18}s^2)\mid s\in\mathcal{S}_i\}.$$

Hence for each $i\in\{0,1,3\}$, we get the family

$$\mathcal{F}_{9,1}^i=\{\chi_{r_{2,5},r_{3,7}}^{a_6^i,a_8,a_9^i,a_{10},a_{18}}\mid a_8,a_{10},a_{18}\in\mathbb{F}_q^\times,(a_6^i,a_9^i)\in\mathcal{A}_i,r_{2,5}\in\{1,\dots,i+1\},r_{3,7}\in\{1,\dots,i+1\}\}$$

of characters of degree $q^2/(i+1)$.

Let $f=2k$. We have that $|\mathcal{A}_3|=(q-1)(q-2)/6$, $|\mathcal{A}_1|=q(q-1)/2$, and $|\mathcal{A}_0|=(q-1)^2/3$. Hence in this case,

- $\mathcal{F}_{9,1}^{f\text{ even},3}$ consists of $8(q-1)^4(q-4)/3$ irreducible characters of degree $q^2/4$,
- $\mathcal{F}_{9,1}^{f\text{ even},1}$ consists of $2q(q-1)^4$ irreducible characters of degree $q^2/2$, and
- $\mathcal{F}_{9,1}^{f\text{ even},0}$ consists of $(q-1)^5/3$ irreducible characters of degree q^2.

Let $f=2k+1$. We have that $|\mathcal{A}_3|=(q-1)(q-2)/6$, $|\mathcal{A}_1|=(q-1)(q-2)/2$, and $|\mathcal{A}_0|=(q-1)(q+1)/3$. Hence in this case,

- $\mathcal{F}_{9,1}^{f\text{ odd},3}$ consists of $8(q-1)^4(q-2)/3$ irreducible characters of degree $q^2/4$,
- $\mathcal{F}_{9,1}^{f\text{ odd},1}$ consists of $2(q-1)^4(q-2)$ irreducible characters of degree $q^2/2$, and
- $\mathcal{F}_{9,1}^{f\text{ odd},0}$ consists of $(q-1)^4(q+1)/3$ irreducible characters of degree q^2.

1.12. **The $[5,9,4]$-core $\mathcal{F}_{9,2}$.** We have that

- $\mathcal{S}=\{\alpha_2,\alpha_3,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_{10},\alpha_{12},\alpha_{18}\}$,
- $\mathcal{Z}=\{\alpha_6,\alpha_8,\alpha_{10},\alpha_{12},\alpha_{18}\}$,
- $\mathcal{A}=\{\alpha_1,\alpha_4\}$ and $\mathcal{L}=\{\alpha_{11},\alpha_{16}\}$,
- $\mathcal{I}=\{\alpha_2,\alpha_5\}$ and $\mathcal{J}=\{\alpha_3,\alpha_7\}$.

The form of Equation (0.1) is

$$\phi(s_3(a_6t_2+a_8t_5)+s_7(a_{18}t_5s_7+a_{10}t_2+a_{12}t_5)=1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases}a_6t_2=a_8t_5\\ a_{10}^2t_2^2+a_{12}^2t_5^2=a_{18}t_5\end{cases}\text{, resp.}\begin{cases}a_6s_3=a_{10}s_7\\ a_8s_3=a_{18}s_7^2+a_{12}s_7\end{cases}.$$

This can be rewritten as

$$\begin{cases}t_2=a_6^{-1}a_8t_5\\ t_5\left((a_6^2a_{12}^2+a_8^2a_{10}^2)t_5+a_6^2a_{18}\right)=0\end{cases}\text{, resp.}\begin{cases}s_3=a_6^{-1}a_{10}s_7\\ s_7(a_{16}a_{18}s_7+a_6a_{12}+a_8a_{10})=0\end{cases}.$$

Let us first assume that $a_{12}=a_8a_{10}/a_6$. Then we have $X'=Y'=1$, and we get the family

$$\mathcal{F}_{9,2}^1=\{\chi^{a_6,a_8,a_{10},a_{18}}\mid a_6,a_8,a_{10},a_{18}\in\mathbb{F}_q^\times\}$$

of $(q-1)^4$ irreducible characters of degree q^2.

Let us then suppose $a_{12} \neq a_8 a_{10}/a_6$. Then we have

$$X'\{x_2(\frac{a_8}{a_6}t)x_5(t) \mid t \in \{0, \frac{a_6^2 a_{18}}{a_6^2 a_{12}^2 + a_8^2 a_{10}^2}\}\} \quad \text{and} \quad Y' = \{x_3(\frac{a_{10}}{a_6}s)x_7(s) \mid s \in \{0, \frac{a_6 a_{12} + a_8 a_{10}}{a_6 a_{18}}\}\}$$

Correspondingly, we get the family

$$\mathcal{F}_{9,2}^2 = \{\chi_{c_{2,5},c_{3,7}}^{a_6,a_8,a_{10},a_{12}^*,a_{18}} \mid a_6, a_8, a_{10}, a_{18} \in \mathbb{F}_q^\times, a_{12}^* \in \mathbb{F}_q^\times \setminus \{a_8 a_{10}/a_6\}, c_{2,5}, c_{3,7} \in \mathbb{F}_2\}$$

of $4(q-1)^4(q-2)$ irreducible characters of degree $q^2/2$.

1.13. **The** $[5, 11, 6]$**-core** $\mathcal{F}_{10}$**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{14}, \alpha_{16}\}$,
- $\mathcal{Z} = \{\alpha_7, \alpha_8, \alpha_{10}, \alpha_{14}, \alpha_{16}\}$,
- $\mathcal{A} = \{\alpha_2\}$ and $\mathcal{L} = \{\alpha_{11}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_4\, \alpha_5\}$ and $\mathcal{J} = \{\alpha_3, \alpha_6, \alpha_9\}$.

The form of Equation (0.1) is

$$\phi\left(s_3(a_7 t_4 + a_8 t_5) + s_6(a_{14} s_6 t_1 + a_8 t_1 + a_{10} t_4) + s_9(a_{16} t_4 s_9 + a_{14} t_5)\right) = 1.$$

Hence X' (resp. Y') are obtained by solving the following system of equations,

$$\begin{cases} a_7 t_4 = a_8 t_5 \\ a_8^2 t_1^2 + a_{10}^2 t_4^2 = a_{14} t_1 \\ a_{14}^2 t_5^2 = a_{16} t_4 \end{cases}, \text{resp.} \begin{cases} a_{14} s_6^2 = a_8 s_6 \\ a_{16} s_9^2 = a_7 s_3 + a_{10} s_6 \\ a_8 s_3 = a_{14} s_9 \end{cases}.$$

We first look at X'. Let $t_5 = 0$. Then $t_4 = 0$ and $t_1 \in \{0, a_8^{-1} a_{14}\}$. Now assume that $t_5 \neq 0$. Then we have that $t_5 = a_7^{-1} a_8 a_{14}^{-1} a_{16}$ and $t_4 = a_7^{-2} a_8^2 a_{14}^{-1} a_{16}$. We have to discuss the solutions of the equation

$$a_8^2 t_1^2 + a_{14} t_1 + a_7^{-4} a_8^4 a_{10}^2 a_{14}^{-4} a_{16}^2 = 0. \tag{1.3}$$

We then look at Y'. Let $s_6 = 0$. Then $(s_3, s_9) \in \{(0,0), (a_7 a_8^{-2} a_{14}^2 a_{16}^{-1}, a_7 a_8^{-1} a_{14} a_{16}^{-1})$. Let us now assume that $s_6 \neq 0$. Then $s_6 = a_8 a_{14}^{-1}$ and $s_3 = a_8^{-1} a_{14} s_9$. We have to discuss the solutions of the equation

$$a_8 a_{14} a_{16} s_9^2 + a_7 a_{14}^2 s_9 + a_8^2 a_{10} = 0. \tag{1.4}$$

Now notice that both Equations (1.3) and (1.4) have 2 solutions if $\mathrm{Tr}((\frac{a_8^3 a_{10} a_{16}}{a_7^2 a_{14}^3})^2) = 0$ and 0 otherwise.

Let us assume that $\mathrm{Tr}(\frac{a_8^6 a_{10}^2 a_{16}^2}{a_7^4 a_{14}^6}) = 0$ (this happens for $(q/2) - 1 = (q-2)/2$ values of $a_{16} \in \mathbb{F}_q^\times$ since $a_{16} \neq 0$ and $t \mapsto t^2$ is invertible). Then we have two values $\bar{t}_1, \bar{t}_2$ satisfying Equation (1.3), and two values $\bar{s}_1, \bar{s}_2$ satisfying Equation (1.4). We get

$$X' = \{1, x_1(\frac{a_{14}}{a_8})\} \cup \{x_1(\bar{t})x_4(a_7^{-2} a_8^2 a_{14}^{-1} a_{16})x_5(a_7^{-1} a_8 a_{14}^{-1} a_{16}) \mid \bar{t} \in \{\bar{t}_1, \bar{t}_2\}\}$$

and

$$Y' = \{1, x_3(a_7 a_8^{-2} a_{14}^2 a_{16}^{-1})x_9(a_7 a_8^{-1} a_{14} a_{16}^{-1})\} \cup \{x_3(\frac{a_{14}\bar{s}}{a_8})x_6(\frac{a_8}{a_{14}})x_9(\bar{s}) \mid \bar{s} \in \{\bar{s}_1, \bar{s}_2\}\}.$$

We get of $8(q-2)(q-1)^4$ irreducible characters of degree $q^3/4$.

Let us now assume that $\mathrm{Tr}(\frac{a_8^6 a_{10}^2 a_{16}^2}{a_7^4 a_{14}^6}) \neq 0$ (this happens for the remaining $q/2$ values of $a_{16} \in \mathbb{F}_q^\times$). Then Equation (1.3) and (1.4) do not have solutions. We get

$$X' = \{1, x_1\left(\frac{a_{14}}{a_8}\right)\} \qquad \text{and} \qquad Y' = \{1, x_3(a_7 a_8^{-2} a_{14}^2 a_{16}^{-1}) x_9(a_7 a_8^{-1} a_{14} a_{16}^{-1})\}.$$

This gives $2q(q-1)^4$ irreducible characters of degree $q^3/2$.

1.14. **The** $[6, 10, 4]$**-core** $\mathcal{F}_{11}$**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{18}\}$,
- $\mathcal{Z} = \{\alpha_6, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{18}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{11}, \alpha_{16}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_5\}$ and $\mathcal{J} = \{\alpha_3, \alpha_7\}$.

The form of Equation (0.1) is

$$\phi(s_3(a_9 t_2^2 + a_6 t_2 + a_8 t_5) + s_7(a_{18} t_5^2 + a_{12} t_5 + a_{10} t_2)) = 1.$$

We have that

$$X' = \{\underline{x}(\underline{t}) \in X \mid a_8 t_5 = a_9 t_2^2 + a_6 t_2 \quad \text{and} \quad a_{10} t_2 = a_{18} t_5^2 + a_{12} t_5\}$$

and

$$Y' = \{\underline{x}(\underline{s}) \in y \mid a_6^2 s_3^2 + a_{10}^2 s_7^2 = a_9 s_3 \quad \text{and} \quad a_{12}^2 s_7^2 + a_8^2 s_3^2 = a_{18} s_7\}.$$

We now focus on the determination of X'. Analogous computations can be carried for determining Y'. We omit the details in the latter case, just mentioning that the cubic equations that show up in the study of X' and Y', which depend on a_i for $i \in \{6, 8, 9, 10, 12, 18\}$, have the same number of solutions for each of the fixed values in $\mathbb{F}_q^\times$ of the a_i's.

Let us fix a_8, a_9 and a_{18} in $\mathbb{F}_q^\times$. By combining the equations defining X', we substitute the value of t_5 as a function of t_2 into the first equation. Let us put $\bar{a}_6 = a_6/a_9$, $\bar{a}_{10} = a_6 a_8 a_{12}/(a_9^2 a_{18})$ and $\bar{a}_{12} = a_8^2 a_{10}/(a_9^2 a_{18})$. Then we get

$$t_2(t_2^3 + (\bar{a}_{12}/\bar{a}_6 + \bar{a}_6^2) t_2 + (\bar{a}_{10} + \bar{a}_{12})) = 0. \tag{1.5}$$

As X' is an abelian subgroup of $X_{\mathcal{S}}$, and Y' is determined in a similar way as previously remarked (in particular, $|X'| = |Y'|$), each choice of a_i, $i \in \{6, 10, 12\}$ such that Equation (1.5) has k solutions yields $k^2(q-1)^3$ irreducible characters of degree q^2/k. The claim follows if we determine the number of solutions of Equation (1.5) for every $\bar{a}_6, \bar{a}_{10}, \bar{a}_{12} \in \mathbb{F}_q^\times$.

Let us first assume that $\bar{a}_{10} = \bar{a}_{12} = \bar{a}_6^3$; this happens for $q-1$ values of $\bar{a}_6, \bar{a}_{10}, \bar{a}_{12} \in \mathbb{F}_q^\times$. In this case, Equation 1.5 is $t_2^4 = 0$ and just has the trivial solution. In this case, we get the family $\mathcal{F}_{11}^1$ as in Table 1.1.

Let us then assume that $\bar{a}_{12} \neq \bar{a}_6^3$ and $a_{10} = a_{12}$; this happens for $(q-1)(q-2)$ values of $\bar{a}_6, \bar{a}_{10}, \bar{a}_{12} \in \mathbb{F}_q^\times$. In this case, Equation 1.5 is $t_2^2(t_2^2 + c) = 0$, where $c = \bar{a}_{10} + \bar{a}_{12} \neq 0$, and we see that its two distinct solutions are zero and the unique square root of c. This gives the family $\mathcal{F}_{11}^2$ as in Table 1.1.

We now assume that $\bar{a}_{12} = \bar{a}_6^3$ and $a_{10} \neq a_{12}$; this happens for $(q-1)(q-2)$ values of $\bar{a}_6, \bar{a}_{10}, \bar{a}_{12} \in \mathbb{F}_q^\times$. Equation 1.5 writes $t_2(t_2^3 + d) = 0$, where $d = \bar{a}_{10} + \bar{a}_6^3 \neq 0$. If $f = 2k+1$, then d has a unique cube root and the equation has two distinct solutions. This gives the family $\mathcal{F}_{11}^{f \text{ odd}, 3}$ as in Table 1.1. Let us then assume that $f = 2k$. We distinguish two cases in turn. We first suppose that $\bar{a}_{10} \in (\bar{a}_6^3 + \mathbb{F}_{q,3}^\times) \setminus \{0\} = \bar{a}_6^3 + \mathbb{F}_{q,3}^\times \setminus \{\bar{a}_6^3\}$; this happens for

$(q-1)((q-1)/3-1 = (q-1)(q-4)/3$ values of $\bar{a}_6, \bar{a}_{10}, \bar{a}_{12} \in \mathbb{F}_q^\times$. In this case, d has three distinct cube roots, and Equation 1.5 has four distinct solutions. This gives the family $\mathcal{F}_{11}^{f \text{ even, } 4}$ as in Table 1.1. Assume then that $\bar{a}_{10} \in (\bar{a}_6^3 + \mathbb{F}_q \setminus \mathbb{F}_{q,3}^\times) \setminus \{0\} = \bar{a}_6^3 + \mathbb{F}_q \setminus \mathbb{F}_{q,3}^\times$; this happens for $2(q-1)^2/3$ values of $\bar{a}_6, \bar{a}_{10}, \bar{a}_{12} \in \mathbb{F}_q^\times$. In this case, d has no cube roots. Therefore, Equation 1.5 only has the solution $t_2 = 0$, which yields the family $\mathcal{F}_{11}^{f \text{ even, } 3}$ as in Table 1.1.

Finally, assume that $\bar{a}_{12} = \bar{a}_6^3$ and $a_{10} \neq a_{12}$. Then we are in the assumptions of Lemma 19 by setting $t = \bar{a}_{12}$, $b = \bar{a}_6$ and $c = \bar{a}_{10}$. We readily get the families $\mathcal{F}_{11}^{f \text{ even, } 5}$ $\mathcal{F}_{11}^{f \text{ even, } 6}$ and $\mathcal{F}_{11}^{f \text{ even, } 7}$ as in Table 1.1 when $f = 2k$, and the families $\mathcal{F}_{11}^{f \text{ odd, } 4}$ $\mathcal{F}_{11}^{f \text{ odd, } 5}$ and $\mathcal{F}_{11}^{f \text{ odd, } 6}$ as in Table 1.1 when $f = 2k+1$, in the cases when

$$t_2^3 + (\bar{a}_{12}/\bar{a}_6 + \bar{a}_6^2)t_2 + (\bar{a}_{10} + \bar{a}_{12}) = 0$$

has 0, 1 or 3 solutions respectively.

Since

$$\mathcal{F}_{11}^{f \text{ even}} = \bigsqcup_{i=1}^{7} \mathcal{F}_{11}^{f \text{ even, } i} \qquad \text{and} \qquad \mathcal{F}_{11}^{f \text{ odd}} = \bigsqcup_{i=1}^{6} \mathcal{F}_{4,2}^{f \text{ odd, } i},$$

the claim is proved.

Form	Freq.	Branch.	Family	Label	Number	Degree
$[2,4,1]$	185	185	$\mathcal{F}_1$	$\chi_{c_2,c_3}^{a_6,a_9}$	$4(q-1)^2$	$q/2$
$[3,10,9]$	1	1	$\mathcal{F}_2^1$	$\chi^{a_8,a_{12},a_{13}}$	$(q-1)^3$	q^3
			$\mathcal{F}_2^2$	$\chi_{c_{1,3,7},c_2}^{a_{5,6,10},a_8,a_{12},a_{13}}$	$4(q-1)^4$	$q^3/2$
$[4,8,2]$	2	2	$\mathcal{F}_3$	$\chi_{c_2,c_3,c_5,c_7}^{a_6,a_9,a_{12},a_{18}}$	$16(q-1)^4$	$q^2/4$
$[4,8,4]$	8	6	$\mathcal{F}_{4,1}$	$\chi_{c_{2,5},c_{3,7}}^{a_6,a_8,a_{10},a_{18}}$	$4(q-1)^4$	$q^2/2$
		2	$\mathcal{F}_{4,2}^{f\text{ even},1}$	$\chi^{a_8,a_9,a_{10},a_{18}^1}$	$2(q-1)^4/3$	q^2
			$\mathcal{F}_{4,2}^{f\text{ even},2}$	$\chi_{d_{2,5},d_{3,7}}^{a_8,a_9,a_{10},a_{18}^2}$	$16(q-1)^4/3$	$q^2/4$
			$\mathcal{F}_{4,2}^{f\text{ odd}}$	$\chi_{c_{2,5},c_{3,7}}^{a_8,a_9,a_{10},a_{18}}$	$4(q-1)^4$	$q^2/2$
$[4,10,5]$	2	2	$\mathcal{F}_5$	$\chi_{c_{1,7},c_{2,6},c_4,c_9}^{a_5,a_8,a_{13},a_{16}}$	$16(q-1)^4$	$q^3/4$
$[4,11,6]$	2	2	$\mathcal{F}_6$	$\chi_{b_{4,7,12,15},c_{2,6,9},c_{4,7,12,15}}^{a_{10},a_{16},a_{19},a_{24}}$	$4q(q-1)^4$	$q^3/2$
$[4,12,9]$	2	1	$\mathcal{F}_{7,1}$	$\chi_{b_{1,5,8,11},b_{4,7,10,13}}^{a_{12},a_{15},a_{19},a_{23}}$	$q^2(q-1)^4$	q^3
		1	$\mathcal{F}_{7,2}^1$	$\chi_{c_{1,4},c_2,c_3}^{a_8,a_{10},a_{11},a_{16}}$	$8(q-1)^4$	$q^4/8$
			$\mathcal{F}_{7,2}^2$	$\chi_{c_{1,4}}^{a_8,a_{10},a_{11},a_{16},e_{6,9}}$	$2(q-1)^4$	$q^4/4$
			$\mathcal{F}_{7,2}^3$	$\chi_{c_2}^{a_8,a_{10},a_{11},a_{16},e_7}$	$2(q-1)^4$	$q^4/4$
			$\mathcal{F}_{7,2}^4$	$\chi_{c_3}^{a_8,a_{10},a_{11},a_{16},e_5}$	$2(q-1)^4$	$q^4/4$
			$\mathcal{F}_{7,2}^5$	$\chi_{c_{1,4}}^{a_8,a_{10},a_{11},a_{16},e_5,e_7}$	$2(q-1)^4$	$q^4/4$
			$\mathcal{F}_{7,2}^6$	$\chi_{c_2}^{a_8,a_{10},a_{11},a_{16},e_5,e_{6,9}}$	$2(q-1)^4$	$q^4/4$
			$\mathcal{F}_{7,2}^7$	$\chi_{c_3}^{a_8,a_{10},a_{11},a_{16},e_{6,9},e_7}$	$2(q-1)^4$	$q^4/4$
			$\mathcal{F}_{7,2}^8$	$\chi_{c_{1,4}}^{a_8,a_{10},a_{11},a_{16},e_5,e_{6,9},e_7}$	$2(q-1)^4$	$q^4/4$
$[5,9,3]$	2	2	$\mathcal{F}_8^1$	$\chi_{c_2,c_3,c_5,c_7}^{a_6,a_9,a_{10}^1,a_{12},a_{18}}$	$8(q-2)(q-1)^4$	$q^2/4$
			$\mathcal{F}_8^2$	$\chi_{c_3,c_5}^{a_6,a_9,a_{10}^2,a_{12},a_{18}}$	$2q(q-1)^4$	$q^2/2$
$[5,9,4]$	4	3	$\mathcal{F}_{9,1}^{f\text{ even},3}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^3,a_8,a_9^3,a_{10},a_{18}}$	$8(q-1)^4(q-4)/3$	$q^2/4$
			$\mathcal{F}_{9,1}^{f\text{ even},1}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^1,a_8,a_9^1,a_{10},a_{18}}$	$2q(q-1)^4$	$q^2/2$
			$\mathcal{F}_{9,1}^{f\text{ even},0}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^0,a_8,a_9^0,a_{10},a_{18}}$	$(q-1)^5/3$	q^2
			$\mathcal{F}_{9,1}^{f\text{ odd},3}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^3,a_8,a_9^3,a_{10},a_{18}}$	$8(q-1)^4(q-2)/3$	$q^2/4$
			$\mathcal{F}_{9,1}^{f\text{ odd},1}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^1,a_8,a_9^1,a_{10},a_{18}}$	$2(q-1)^4(q-2)$	$q^2/2$
			$\mathcal{F}_{9,1}^{f\text{ odd},0}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^1,a_8,a_9^1,a_{10},a_{18}}$	$(q-1)^4(q+1)/3$	q^2
		1	$\mathcal{F}_{9,2}^1$	$\chi^{a_6,a_8,a_{10},a_{18}}$	$(q-1)^4$	q^2
			$\mathcal{F}_{9,2}^2$	$\chi_{c_{2,5},c_{3,7}}^{a_6,a_8,a_{10},a_{12}^*,a_{18}}$	$4(q-1)^4(q-2)$	$q^2/2$
$[5,11,6]$	2	2	$\mathcal{F}_{10}^1$	$\chi_{c_{1,3},c_{4,5},c_6,c_9}^{a_7,a_8,a_{10}^1,a_{14},a_{16}}$	$8(q-2)(q-1)^4$	$q^3/4$
			$\mathcal{F}_{10}^2$	$\chi_{c_{1,3},c_9}^{a_7,a_8,a_{10}^2,a_{14},a_{16}}$	$2q(q-1)^4$	$q^3/2$
$[6,10,4]$	1	1	$\mathcal{F}_{11}^1$	$\chi^{a_9,a_{10},a_{12},a_{18}}$	$(q-1)^4$	q^2
			$\mathcal{F}_{11}^2$	$\chi_{b_{2,5},b_{3,7}}^{a_8^*,a_9,a_{10},a_{12},a_{18}}$	$4(q-1)^4(q-2)$	$q^2/2$
			$\mathcal{F}_{11}^{f\text{ even},3}$	$\chi^{a_6^{*1},a_9,a_{10},a_{12},a_{18}}$	$2(q-1)^5/3$	q^2
			$\mathcal{F}_{11}^{f\text{ even},4}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^{*2},a_9,a_{10},a_{12},a_{18}}$	$16(q-1)^4(q-4)/3$	$q^2/4$
			$\mathcal{F}_{11}^{f\text{ even},5}$	$\chi^{a_6^0,a_8^0,a_9,a_{10},a_{12},a_{18}}$	$(q-1)^5(q-2)/3$	q^2
			$\mathcal{F}_{11}^{f\text{ even},6}$	$\chi_{c_{2,5},c_{3,7}}^{a_6^1,a_8^1,a_9,a_{10},a_{12},a_{18}}$	$2q(q-1)^4(q-3)$	$q^2/2$
			$\mathcal{F}_{11}^{f\text{ even},7}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^3,a_8^3,a_9,a_{10},a_{12},a_{18}}$	$8(q-1)^4(q-4)(q-5)/3$	$q^2/4$
			$\mathcal{F}_{11}^{f\text{ odd},3}$	$\chi_{b_{2,5},b_{3,7}}^{a_6^1,a_9,a_{10},a_{12},a_{18}}$	$4(q-1)^4(q-2)$	$q^2/2$
			$\mathcal{F}_{11}^{f\text{ odd},4}$	$\chi^{a_6^0,a_8^0,a_9,a_{10},a_{12},a_{18}}$	$(q-1)^4(q-2)(q+1)/3$	q^2
			$\mathcal{F}_{11}^{f\text{ odd},5}$	$\chi_{c_{2,5},c_{3,7}}^{a_6^1,a_8^1,a_9,a_{10},a_{12},a_{18}}$	$2(q-1)^4(q-2)(q-3)$	$q^2/2$
			$\mathcal{F}_{11}^{f\text{ odd},6}$	$\chi_{d_{2,5},d_{3,7}}^{a_6^3,a_8^3,a_9,a_{10},a_{12},a_{18}}$	$8(q-1)^4(q-2)(q-5)/3$	$q^2/4$

TABLE 1.1. The irreducible characters arising from the nonabelian cores of $\mathrm{UF}_4(q)$, where $q = 2^f$