COMPUTATIONAL DETAILS FOR THE ANALYSIS OF THE NONABELIAN CORES IN TYPES D_6 AND E_6

TUNG LE, KAY MAGAARD AND ALESSANDRO PAOLINI

This file collects the full-detailed computations about the parametrization of the sets $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ for each pair $(\mathcal{S},\mathcal{Z})$ associated to a $[z,m,c]$-core as in the paper *The irreducible characters of the Sylow p-subgroups of the Chevalley groups* $\mathrm{D}_6(p^f)$ *and* $\mathrm{E}_6(p^f)$. The notation used here is the same as in Section 5 of the work.

1. The heartless cores in D_6 and E_6

The $[3,9,6]$-core in E_6. We have that

- $\mathcal{S}=\{\alpha_7,\alpha_{11},\alpha_{12},\alpha_{16},\alpha_{19},\alpha_{23},\alpha_{24},\alpha_{29},\alpha_{31}\}$,
- $\mathcal{Z}=\{\alpha_{23},\alpha_{29},\alpha_{31}\}$,
- $\mathcal{A}=\{\alpha_3,\alpha_4,\alpha_5,\alpha_9,\alpha_{10},\alpha_{15},\alpha_{18},\alpha_{21},\}$ and $\mathcal{L}=\{\alpha_2,\alpha_6,\alpha_{14},\alpha_{20},\alpha_{22},\alpha_{25},\alpha_{26},\alpha_{28}\}$,
- $\mathcal{I}=\{\alpha_7,\alpha_{11},\alpha_{19}\}$ and $\mathcal{J}=\{\alpha_{12},\alpha_{16},\alpha_{24}\}$.

The form of Equation (4.3) is

$$s_{12}(a_{23}t_{11}-a_{29}t_{19})+s_{16}(-a_{23}t_7+a_{31}t_{19})+s_{24}(a_{29}t_7+a_{31}t_{11})=0.$$

If $p\neq 2$, then $X'=Y'=1$. We obtain the family $\mathcal{F}_1^{p\neq 2}$ in Table 5.2.

If $p=2$, we have that

$$X':=\{x_{7,11,19}(t)\mid t\in\mathbb{F}_q\}\quad\text{and}\quad Y':=\{x_{12,16,24}(s)\mid s\in\mathbb{F}_q\},$$

where for every $s,t\in\mathbb{F}_q$,

$$x_{7,11,19}(t):=x_7(a_{31}t)x_{11}(a_{29}t)x_{19}(a_{23}t)\quad\text{and}\quad x_{12,16,24}(s):=x_{12}(a_{31}s)x_{16}(a_{29}s)x_{24}(a_{23}s),$$

and we get the family $\mathcal{F}_1^{p=2}$ in Table 5.2.

The $[4,8,4]$-core in E_6. We have that

- $\mathcal{S}=\{\alpha_2,\alpha_4,\alpha_7,\alpha_8,\alpha_{10},\alpha_{12},\alpha_{14},\alpha_{18}\}$,
- $\mathcal{Z}=\{\alpha_8,\alpha_{12},\alpha_{14},\alpha_{18}\}$,
- $\mathcal{A}=\{\alpha_3,\alpha_5,\alpha_9,\alpha_{15}\}$ and $\mathcal{L}=\{\alpha_6,\alpha_{11},\alpha_{13},\alpha_{16}\}$,
- $\mathcal{I}=\{\alpha_2,\alpha_7\}$ and $\mathcal{J}=\{\alpha_4,\alpha_{10}\}$.

The form of Equation (4.3) is

$$s_4(-a_8t_2-a_{12}t_7)+s_{10}(-a_{14}t_2-a_{18}t_7)=0$$

If $a_{18}^*:=a_{18}\neq a_{12}a_{14}/a_8$, then $X'=Y'=1$. We obtain the family $\mathcal{F}_3^1$ in Table 5.2. If $a_{18}=a_{12}a_{14}/a_8$, we have that

$$X':=\{x_{2,7}(t)\mid t\in\mathbb{F}_q\}\quad\text{and}\quad Y':=\{x_{4,10}(s)\mid s\in\mathbb{F}_q\},$$

where for every $s,t\in\mathbb{F}_q$,

$$x_{2,7}(t):=x_2(a_{12}t)x_7(-a_8t)\quad\text{and}\quad x_{4,10}(s):=x_4(a_{14}s)x_{10}(-a_8s),$$

and we get the family $\mathcal{F}_3^2$ in Table 5.2.

The $[5,10,5]$**-core in** E_6**.** We have that

- $\mathcal{S}=\{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_7,\alpha_9,\alpha_{10},\alpha_{13},\alpha_{17},\alpha_{19}\}$,
- $\mathcal{Z}=\{\alpha_7,\alpha_9,\alpha_{10},\alpha_{17},\alpha_{19}\}$,
- $\mathcal{A}=\{\alpha_2,\alpha_8,\alpha_{14}\}$ and $\mathcal{L}=\{\alpha_6,\alpha_{11},\alpha_{16}\}$,
- $\mathcal{I}=\{\alpha_1,\alpha_4,\alpha_{13}\}$ and $\mathcal{J}=\{\alpha_3,\alpha_5\}$.

The form of Equation (4.3) is

$$s_1(t_3a_7)+s_4(-t_3a_9+t_5a_{10})+s_{13}(t_5a_{19})=0.$$

We have that $X':=\{x_{1,4,13}(t)\mid t\in\mathbb{F}_q\}$ and $Y'=1$, where for every $t\in\mathbb{F}_q$,

$$x_{1,4,13}(t)=x_1(a_9a_{19}t)x_4(a_7a_{19}t)x_{13}(-a_7a_{10}t).$$

We get the family $\mathcal{F}_4$ in Table 5.2.

The $[5,12,8]$**-core of** E_6**.** We have that

- $\mathcal{S}=\{\alpha_3,\alpha_4,\alpha_6,\alpha_9,\alpha_{10},\alpha_{11},\alpha_{15},\alpha_{16},\alpha_{17},\alpha_{22},\alpha_{26},\alpha_{27}\}$,
- $\mathcal{Z}=\{\alpha_9,\alpha_{15},\alpha_{16},\alpha_{26},\alpha_{27}\}$,
- $\mathcal{A}=\{\alpha_1,\alpha_2,\alpha_5,\alpha_7,\alpha_8,\alpha_{14}\}$ and $\mathcal{L}=\{\alpha_{12},\alpha_{13},\alpha_{18},\alpha_{20},\alpha_{23},\alpha_{25}\}$,
- $\mathcal{I}=\{\alpha_3,\alpha_4,\alpha_6,\alpha_{17}\}$ and $\mathcal{J}=\{\alpha_{10},\alpha_{11},\alpha_{22}\}$.

The form of Equation (4.3) is

$$s_3(a_{15}t_{10})+s_4(a_{16}t_{11}+a_{26}t_{22})+s_6(-a_{16}t_{10}-a_{27}t_{22})+s_{17}(a_{27}t_{11}+a_{26}t_{10})=0.$$

We have that $Y'=1$ and $X'=\{x_{3,4,6,17}(t)\mid t\in\mathbb{F}_q\}$, where for every $t\in\mathbb{F}_q$,

$$x_{3,4,6,17}(t):=x_3(2a_{16}a_{26}t)x_4(a_{15}a_{27}t)x_6(a_{15}a_{26}t)x_{17}(-a_{15}a_{16}t).$$

We get the family $\mathcal{F}_5$ in Table 5.2.

The $[5,15,11]$**-core of** E_6**.** We have that

- $\mathcal{S}=\{\alpha_1,\alpha_4,\alpha_6,\alpha_7,\alpha_9,\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{14},\alpha_{16},\alpha_{19},\alpha_{22},\alpha_{24},\alpha_{25}\}$,
- $\mathcal{Z}=\{\alpha_{12},\alpha_{16},\alpha_{22},\alpha_{24},\alpha_{25}\}$,
- $\mathcal{A}=\{\alpha_2,\alpha_3,\alpha_5,\alpha_{15}\}$ and $\mathcal{L}=\{\alpha_8,\alpha_{17},\alpha_{20},\alpha_{21}\}$.
- $\mathcal{I}=\{\alpha_1,\alpha_4,\alpha_6,\alpha_{13},\alpha_{14}\}$ and $\mathcal{J}=\{\alpha_7,\alpha_9,\alpha_{10},\alpha_{11},\alpha_{19}\}$.

The form of Equation (4.3) is

$$s_7(a_{12}t_4+a_{22}t_{14})+s_9(-a_{12}t_1-a_{24}t_{14})+s_{10}(a_{16}t_6-a_{24}t_{13})+\\ s_{11}(-a_{16}t_4-a_{25}t_{13})+s_{19}(-a_{22}t_1-a_{24}t_4+a_{25}t_6)=0.$$

If $p\neq 3$, we have $X'=Y'=1$, and we get the family $\mathcal{F}_6^{p\neq 3}$ in Table 5.2.

If $p=3$, we have that

$$X'=\{x_{1,4,6,13,14}(t)\mid t\in\mathbb{F}_q\},\quad\text{and}\quad Y'=\{x_{7,9,10,11,19}(s)\mid s\in\mathbb{F}_q\},$$

where for every $s,t\in\mathbb{F}_q$,

$$x_{1,4,6,13,14}(t)=x_1(-a_{24}a_{25}t)x_4(-a_{22}a_{25}t)x_6(a_{22}a_{24}t)x_{13}(a_{16}a_{22}t)x_{14}(a_{12}a_{25}t)$$

and

$$x_{7,9,10,11,19}(s)=x_7(a_{16}a_{24}s)x_9(a_{16}a_{22}s)x_{10}(a_{12}a_{25}s)x_{11}(-a_{12}a_{24}s)x_{19}(-a_{12}a_{16}s).$$

We get the family $\mathcal{F}_6^{p=3}$ in Table 5.2.

The $[6,12,6]$**-core in** E_6**.** We have that

- $\mathcal{S}=\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_8,\alpha_9,\alpha_{10},\alpha_{12},\alpha_{15},\alpha_{21},\alpha_{23},\alpha_{25}\}$,

- $\mathcal{Z} = \{\alpha_8, \alpha_{10}, \alpha_{12}, \alpha_{15}, \alpha_{23}, \alpha_{25}\}$,
- $\mathcal{A} = \{\alpha_3, \alpha_6, \alpha_7, \alpha_{11}\}$ and $\mathcal{L} = \{\alpha_{13}, \alpha_{16}, \alpha_{19}, \alpha_{20}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_5\}$ and $\mathcal{J} = \{\alpha_4, \alpha_9, \alpha_{21}\}$.

The form of Equation (4.3) is

$$s_4(-a_8t_2 + a_{10}t_5) + s_9(-a_{12}t_1 + a_{15}t_5) + s_{21}(-a_{23}t_1 - a_{25}t_2) = 0.$$

If $a_{25}^* := a_{25} \neq -a_8a_{15}a_{23}/(a_{10}a_{12})$, then we have $X' = Y' = 1$. This gives the family $\mathcal{F}_{10}^1$ in Table 5.2. If $a_{25} = -a_8a_{15}a_{23}/(a_{10}a_{12})$, then we get

$$X' := \{x_{1,2,5}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{4,9,21}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{2,7}(t) := x_1(a_8a_{15}t)x_2(a_{10}a_{15}t)x_5(a_8a_{12}t) \quad \text{and} \quad x_4(a_{15}a_{23}s)x_9(-a_{10}a_{23}s)x_{21}(a_{10}a_{12}s).$$

We get the family $\mathcal{F}_{10}^2$ in Table 5.2.

The $[6, 13, 7]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_{11}, \alpha_{12}, \alpha_{14}, \alpha_{18}, \alpha_{19}, \alpha_{24}, \alpha_{26}, \alpha_{28}\}$,
- $\mathcal{Z} = \{\alpha_7, \alpha_{11}, \alpha_{18}, \alpha_{19}, \alpha_{26}, \alpha_{28}\}$,
- $\mathcal{A} = \{\alpha_4, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{13}, \alpha_{15}\}$ and $\mathcal{L} = \{\alpha_2, \alpha_{16}, \alpha_{17}, \alpha_{20}, \alpha_{21}, \alpha_{25}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_{14}, \alpha_{24}\}$ and $\mathcal{J} = \{\alpha_3, \alpha_6, \alpha_{12}\}$.

The form of Equation (4.3) is

$$s_1(a_7t_3) + s_5(-a_{18}t_{12} + a_{11}t_6), +s_{14}(-a_{19}t_3 + a_{26}t_{12}) + s_{24}(a_{28}t_6) = 0.$$

We have that $X' := \{x_{1,5,14,24}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = 1$, where for $t \in \mathbb{F}_q$,

$$x_{1,5,14,24} := x_1(a_{18}a_{19}a_{28}t)x_5(a_7a_{26}a_{28}t)x_{14}(a_7a_{18}a_{28}t)x_{24}(-a_7a_{11}a_{26}t)$$

We get the family $\mathcal{F}_{11}$ in Table 5.2.

The $[6, 14, 8]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{Z} = \{\alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{16}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_5\}$ and $\mathcal{L} = \{\alpha_{17}, \alpha_{18}, \alpha_{19}\}$,
- $\mathcal{I} = \{\alpha_3, \alpha_6, \alpha_7, \alpha_{11}\}$ and $\mathcal{J} = \{\alpha_4, \alpha_8, \alpha_{10}, \alpha_{14}\}$.

The form of Equation (4.3) is

$$s_4(-a_{12}t_7 + a_{16}t_{11}) + +s_8(-a_{13}t_3 + a_{20}t_{11}) + s_{10}(-a_{15}t_3 + a_{16}t_6) + s_{14}(a_{20}t_6 - a_{22}t_7) = 0.$$

If $a_{22}^* := a_{22} \neq a_{12}a_{15}a_{20}^2/(a_{13}a_{16}^2)$, then we have $X' = Y' = 1$. We get the family $\mathcal{F}_{12}^1$ in Table 5.2. If $a_{22} = a_{12}a_{15}a_{20}^2/(a_{13}a_{16}^2)$, then we have that

$$X' := \{x_{3,6,7,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{4,8,10,14}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{3,6,7,11}(t) := x_3(a_{12}a_{16}a_{20}t)x_6(a_{12}a_{15}a_{20}t)x_7(a_{13}a_{16}^2t)x_{11}(a_{12}a_{13}a_{16}t)$$

and

$$x_{4,8,10,14}(s) := x_4(a_{15}a_{20}^2s)x_8(-a_{15}a_{16}a_{20}s)x_{10}(a_{13}a_{16}a_{20}s)x_{14}(-a_{13}a_{16}^2s).$$

This gives the family $\mathcal{F}_{12}^2$ in Table 5.2.

The $[6, 16, 12]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{16}, \alpha_{18}, \alpha_{19}, \alpha_{22}, \alpha_{24}, \alpha_{25}\}$,
- $\mathcal{Z} = \{\alpha_{12}, \alpha_{16}, \alpha_{18}, \alpha_{22}, \alpha_{24}, \alpha_{25}\}$,

- $\mathcal{A} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_{15}\}$ and $\mathcal{L} = \{\alpha_8, \alpha_{17}, \alpha_{20}, \alpha_{21}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_{13}, \alpha_{14}\}$ and $\mathcal{J} = \{\alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{19}\}$.

The form of Equation (4.3) is

$$s_1(a_{22}t_{19} + a_{12}t_9) + s_4(a_{16}t_{11} + a_{24}t_{19}) + s_6(-a_{16}t_{10} - a_{25}t_{19}) + s_7(a_{18}t_{10}) + s_{13}(a_{24}t_{10} + a_{25}t_{11}) + s_{14}(a_{24}t_9) = 0.$$

We have $X' := X_1'X_2'$ with $X_1' := \{x_{1,6,7,14}(t_1) \mid t_1 \in \mathbb{F}_q\}$ and $X_2 := \{x_{4,6,7,13}(t_2) \mid t_2 \in \mathbb{F}_q\}$, and $Y' = 1$, where

$$x_{1,6,7,14}(t_1) := x_1(a_{18}a_{24}a_{25}t_1)x_6(a_{18}a_{22}a_{24}t_1)x_7(a_{16}a_{22}a_{24}t_1)x_{14}(-a_{12}a_{18}a_{25}t_1)$$

and

$$x_{4,6,7,13}(t_2) := x_4(a_{18}a_{25}t_2)x_6(a_{18}a_{24}t_2)x_7(2a_{16}a_{24}t_2)x_{13}(-a_{16}a_{18}t_2).$$

We notice that each of X_1' and X_2' are subgroups, but we have

$$[x_{1,6,7,14}(t_1), x_{4,6,7,13}(t_2)] = x_{12}(a_{16}a_{18}a_{22}a_{24}a_{25}t_1t_2)x_{16}(2a_{12}a_{18}a_{22}a_{24}a_{25}t_1t_2),$$

hence

$$\lambda([x_{1,6,7,14}(t_1), x_{4,6,7,13}(t_2)]) = \phi(3a_{12}a_{16}a_{18}a_{22}a_{24}a_{25})$$

and X' is not necessarily a subgroup of $X_{\mathcal{S}}$.

If $p \neq 3$, then we can apply again the Reduction Lemma with arm X_1' and leg X_2', reducing to the abelian subquotient $X_{\mathcal{Z}}/(\ker\lambda)$. This gives the family $\mathcal{F}_{14}^{p\neq 3}$ in Table 5.2.

If $p = 3$, then X' and $X'X_{\mathcal{Z}}$ are abelian subgroups of $X_{\mathcal{S}}/(\ker\lambda)$. In this case, we obtain the family $\mathcal{F}_{14}^{p=3}$ in Table 5.2.

The $[7, 15, 9]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{Z} = \{\alpha_9, \alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{16}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_5\}$ and $\mathcal{L} = \{\alpha_{17}, \alpha_{18}, \alpha_{19}\}$,
- $\mathcal{I} = \{\alpha_3, \alpha_6, \alpha_7, \alpha_{11}\}$ and $\mathcal{J} = \{\alpha_4, \alpha_8, \alpha_{10}, \alpha_{14}\}$.

The form of Equation (4.3) is

$$s_4(-a_9t_3 - a_{12}t_7 + a_{16}t_{11}) + s_8(-a_{13}t_3 + a_{20}t_{11}) + s_{10}(-a_{15}t_3 + a_{16}t_6) + s_{14}(a_{20}t_6 - a_{22}t_7) = 0.$$

If $a_{20} = a_{13}a_{16}/a_9$, or if $a_{20}^* := a_{20} \neq a_{13}a_{16}/a_9$ and $a_{22}^* := a_{22} \neq -a_{12}a_{15}a_{20}^2/(a_9a_{16}a_{20} - a_{13}a_{16}^2)$, then we have that $X' = Y' = 1$. In these case we get, respectively, the family $\mathcal{F}_{16}^2$ and the family $\mathcal{F}_{16}^1$ in Table 5.2. If $a_{20}^* := a_{20} \neq a_{13}a_{16}/a_9$ and $a_{22} = -a_{12}a_{15}a_{20}^2/(a_9a_{16}a_{20} - a_{13}a_{16}^2)$, then we have that

$$X' := \{x_{3,6,7,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{4,8,10,14}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{3,6,7,11}(t) := x_3(a_{12}a_{16}a_{20}t)x_6(a_{12}a_{15}a_{20}t)x_7((a_{13}a_{16} - a_9a_{20})a_{16}t)x_{11}(a_{12}a_{13}a_{16}t)$$

and

$$x_{4,8,10,14}(s) := x_4(a_{15}a_{20}^2s)x_8(-a_{15}a_{16}a_{20}s)x_{10}((a_{13}a_{16} - a_9a_{20})a_{20}s)x_{14}((a_9a_{20} - a_{13}a_{16})a_{16}s).$$

In this case, we get the family $\mathcal{F}_{16}^3$ in Table 5.2.

2. The cores with a heart in D_6 and E_6

The $[4,18,18]$-core in D_6. We have that

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{21}, \alpha_{22}, \alpha_{28}\}$,
- $\mathcal{Z} = \{\alpha_{16}, \alpha_{21}, \alpha_{22}, \alpha_{28}\}$,
- $\mathcal{A} = \{\alpha_3, \alpha_8, \alpha_9, \alpha_{13}\}$ and $\mathcal{L} = \{\alpha_{20}, \alpha_{23}, \alpha_{24}, \alpha_{26}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_6, \alpha_{12}, \alpha_{14}, \alpha_{15}\}$ and $\mathcal{J} = \{\alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{17}, \alpha_{18}, \alpha_{19}\}$.

The form of Equation (4.3) is

$$s_7(a_{21}t_{14} + a_{22}t_{15}) + s_{10}(-a_{16}t_6 - a_{21}t_{12}) + s_{11}(-a_{16}t_5 - a_{22}t_{12}) + s_{17}(a_{21}t_5 + a_{22}t_6) + \\ + s_{18}(-a_{21}t_1 - a_{28}t_{15}) + s_{19}(-a_{22}t_1 - a_{28}t_{14}) = 0.$$

If $p \neq 2$, then $X' = Y' = 1$, and $\overline{V} = X_2X_4Z/(\ker\lambda)$ is abelian. We obtain the family $\mathcal{F}_3^{p\neq 2}$ in Table 5.1.

If $p = 2$, we have that $X' := X'_1X'_2$ and $Y'_1Y'_2$, where

$$X'_1 := \{x_{1,14,15}(t_1) \mid t_1 \in \mathbb{F}_q\} \quad \text{and} \quad X'_1 := \{x_{5,6,12}(t_2) \mid t_2 \in \mathbb{F}_q\},$$
$$Y'_1 := \{x_{7,18,19}(s_1) \mid s_1 \in \mathbb{F}_q\} \quad \text{and} \quad Y'_2 := \{x_{10,11,17}(s_2) \mid s_2 \in \mathbb{F}_q\}$$

and for every $s_1, s_2, t_1, t_2 \in \mathbb{F}_q$,

$$x_{1,14,15}(t_1) := x_1(a_{28}t_1)x_{14}(a_{22}t_1)x_{15}(a_{21}t_1) \quad \text{and} \quad x_{5,6,12}(t_2) := x_5(a_{22}t_2)x_6(a_{21}t_2)x_{12}(a_{16}t_2)$$
$$x_{7,18,19}(s_1) := x_7(a_{28}s_1)x_{18}(a_{22}s_1)x_{19}(a_{21}s_1) \quad \text{and} \quad x_{10,11,17}(s_2) := x_{10}(a_{22}s_2)x_{11}(a_{21}s_2)x_{17}(a_{16}s_2).$$

Notice that X' is a subgroup of $\overline{V}$. We extend λ to $\lambda' = \lambda^{c_{7,18,19}, c_{10,11,17}}$ for every $c_{7,18,19}, c_{10,11,17} \in \mathbb{F}_q$. In $\overline{V}$, we have that $[X'_1, X_4] = [X'_2, X_2] = 1$, and that

$$[x_2(s_2)x_4(s_4), x_{1,14,15}(t_1)x_{5,6,12}(t_2)] = x_{7,18,19}(s_2t_1)x_{10,11,17}(s_4t_2)x_{16}(a_{21}a_{22}s_4t_2^2)\cdot \\ \cdot x_{21}(a_{22}a_{28}s_2t_1^2 + a_{16}a_{22}s_4t_2^2)x_{22}(a_{21}a_{28}s_2t_1^2 + a_{16}a_{21}s_4t_2^2)x_{28}(a_{21}a_{22}s_2t_1^2).$$

We then want to apply the Reduction Lemma with X' as a candidate for an arm, and X_2X_4 as a candidate for a leg. We apply λ to the above, and we use Remark 5.1 study the equation

$$\phi(s_2t_1(c_{7,18,19} + a_{21}a_{22}a_{28}t_1) + s_4t_2(c_{10,11,17} + a_{16}a_{21}a_{22}t_2)) = 1.$$

If $c_{7,18,19} = 0$ and $c_{10,11,17} = 0$, then $X'^{,2}_{(0,0)} = Y'^{,2}_{(0,0)} = 1$ and $V^2_{(0,0)}$ is abelian. This gives the family $\mathcal{F}_3^{1,p=2}$ in Table 5.1. If $a_{7,18,19} := c_{7,18,19} \neq 0$ and $c_{10,11,17} = 0$, then

$$X'^{,2}_{(a_{7,18,19},0)} := \{1, x_{1,14,15}(a_{7,18,19}/(a_{21}a_{22}a_{28}))\} \quad \text{and} \quad Y'^{,2}_{(a_{7,18,19},0)} := \{1, x_2(a_{21}a_{22}a_{28}/(a_{7,18,19}^2))\},$$

and $V^2_{(a_{7,18,19},0)}$ is abelian. This gives the family $\mathcal{F}_3^{2,p=2}$ in Table 5.1. If $c_{7,18,19} = 0$ and $a_{10,11,17} := c_{10,11,17} = 0$, then

$$X'^{,2}_{(0,a_{10,11,17})} := \{1, x_{5,6,12}(a_{10,11,17}/(a_{16}a_{21}a_{22}))\} \quad \text{and} \quad Y'^{,2}_{(0,a_{10,11,17})} := \{1, x_4(a_{16}a_{21}a_{22}/(a_{10,11,17}^2))\},$$

and $V^2_{(0,a_{10,11,17})}$ is abelian. This gives the family $\mathcal{F}_3^{3,p=2}$ in Table 5.1. Finally, if $a_{7,18,19} := c_{7,18,19} \neq 0$ and $a_{10,11,17} := c_{10,11,17} \neq 0$, then we have that $X'^{,2}_{(a_{7,18,19},a_{10,11,17})} = X'^{,2}_{(a_{7,18,19},0)}X'^{,2}_{(0,a_{10,11,17})}$ and $Y'^{,2}_{(a_{7,18,19},a_{10,11,17})} = Y'^{,2}_{(a_{7,18,19},0)}Y'^{,2}_{(0,a_{10,11,17})}$, and $V^2_{(a_{7,18,19},a_{10,11,17})}$ is abelian. This yields the family $\mathcal{F}_3^{4,p=2}$ in Table 5.1.

The $[4,21,28]$-core in D_6. We have that

- $\mathcal{S} = \{\alpha_1, \alpha_3\} \cup \{\alpha_5, \ldots, \alpha_{22}\} \cup \{\alpha_{26}\}$,

- $\mathcal{Z} = \{\alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{26}\}$,
- $\mathcal{A} = \{\alpha_2, \alpha_4\}$ and $\mathcal{L} = \{\alpha_{23}, \alpha_{24}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}\}$ and $\mathcal{J} = \{\alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{17}, \alpha_{18}, \alpha_{19}\}$.

The form of Equation (4.3) is

$$s_{12}(a_{21}t_{10} + a_{22}t_{11}) + s_{14}(-a_{20}t_6 - a_{21}t_7) + s_{15}(-a_{20}t_5 - a_{22}t_7) + s_{17}(a_{21}t_5 + a_{22}t_6) + \\ + s_{18}(-a_{21}t_1 - a_{26}t_{11}) + s_{19}(-a_{22}t_1 - a_{26}t_{10}) = 0.$$

Let $p \neq 2$. Then $X' = Y' = 1$, and $\overline{V} = X_3X_8X_9X_{13}X_{16}Z/(\ker\lambda)$. Notice that X_8 and X_9 are in direct product in $\overline{V}$, that $[X_3, X_{13}] = 1$, and that

$$\lambda([x_{16}(s_{16}), x_3(t_3)x_{13}(t_{13})]) = \lambda(x_{26}(s_{16}t_{13})x_{20}(-s_{16}t_3)) = \phi(a_{26}s_{16}t_{13} - a_{20}s_{16}t_3).$$

Hence we apply again the Reduction Lemma with X_3X_{13} as a candidate for an arm and X_{16} as a candidate for a leg, and get $X^{',2} = \{x_{3,13}(t) \mid t \in \mathbb{F}_q\}$ and $Y^{',2} = 1$, where $x_{3,13}(t) := x_3(a_{26}t)x_{13}(a_{20}t)$ for every $t \in \mathbb{F}_q$. The subquotient $V^2 = Y''X_8X_9Z/(\ker\lambda)$ is abelian. Hence we get the family $\mathcal{F}_4^{p\neq 2}$ in Table 5.1.

Let $p = 2$. Then we have that $X' := X'_1X'_2$ and $Y'_1Y'_2$, where

$$X'_1 := \{x_{1,10,11}(t_1) \mid t_1 \in \mathbb{F}_q\} \quad \text{and} \quad X'_1 := \{x_{5,6,7}(t_2) \mid t_2 \in \mathbb{F}_q\},$$
$$Y'_1 := \{x_{12,18,19}(s_1) \mid s_1 \in \mathbb{F}_q\} \quad \text{and} \quad Y'_2 := \{x_{14,15,17}(s_2) \mid s_2 \in \mathbb{F}_q\}$$

and for every $s_1, s_2, t_1, t_2 \in \mathbb{F}_q$,

$$x_{1,10,11}(t_1) := x_1(a_{26}t_1)x_{10}(a_{22}t_1)x_{11}(a_{21}t_1) \quad \text{and} \quad x_{5,6,7}(t_2) := x_5(a_{22}t_2)x_6(a_{21}t_2)x_7(a_{20}t_2)$$

$x_{12,18,19}(s_1) := x_{12}(a_{26}s_1)x_{18}(a_{22}s_1)x_{19}(a_{21}s_1)$ and $x_{14,15,17}(s_2) := x_{14}(a_{22}s_2)x_{15}(a_{21}s_2)x_{17}(a_{20}s_2)$. We have that X' is a subgroup of $\overline{V} = X_3X_8X_9X_{13}X_{16}X'Y'Z/(\ker\lambda)$. We notice that we have the same commutator relations as after the first reduction for $p \neq 2$ between X_3, X_{13} and X_{16}. Moreover, we notice that $[X', X_{16}] = [X_8, X_{16}] = [X_9, X_{16}] = 1$, and that $X_3 \cap [\overline{V}, \overline{V}] = 1 = X_{13} \cap [\overline{V}, \overline{V}]$. The Reduction Lemma also applies in this case with X_3X_{13} as a candidate for an arm and X_{16} as a candidate for a leg, and get $X^{',2} = \{x_{3,13}(t) \mid t \in \mathbb{F}_q\}$ and $Y^{',2} = 1$ as before, reducing to the subquotient $V^2 = X_{3,13}X_8X_9X'Y'Z/(\ker\lambda)$.

We now extend λ to $\lambda' = \lambda^{c_{12,18,19}, c_{14,15,17}}$ for every $c_{12,18,19}, c_{14,15,17} \in \mathbb{F}_q$. In V^2, we have

$$[x_8(s_8)x_9(s_9)x_{3,13}(s_1), x_{1,10,11}(t_1)x_{5,6,7}(t_2)] = x_{12,18,19}(a_{20}s_1t_2 + s_8t_1)x_{14,15,17}(a_{26}s_1t_1 + s_9t_2) \cdot \\ \cdot\ x_{21}(a_{20}a_{22}s_9t_2^2 + a_{22}a_{26}s_8t_1^2)x_{20}(a_{21}a_{22}s_9t_2^2)x_{22}(a_{20}a_{21}s_9t_2^2 + a_{21}a_{26}s_8t_1^2)x_{26}(a_{21}a_{22}s_8t_1^2),$$

We can take X' and $X_{3,13}X_8X_9$ as candidates for arm and leg respectively. Applying λ to the above, we use Remark 5.1 to study the equation

$$\phi(s_1(c_{12,18,19}a_{20}t_2 + c_{14,15,17}a_{26}t_1) + s_8(c_{12,18,19}t_1 + a_{21}a_{22}a_{26}t_1^2) + s_9(c_{14,15,17}t_2 + a_{20}a_{21}a_{22}t_2^2)) = 1.$$

If $c_{12,18,19} = c_{14,15,17} = 0$, then $X^{',3}_{(0,0)} = 1$ and $Y^{',3}_{(0,0)} = X_{3,13}$, and $V^3_{(0,0)} = X_{3,13}Z/(\ker\lambda)$ is abelian. This gives the family $\mathcal{F}_4^{1,p=2}$ in Table 5.1.

Let us denote by $\omega_{21,22}$ the unique square root of $a_{21}a_{22}$. If $a_{12,18,19} := c_{12,18,19} \neq 0$ and $c_{14,15,17} = 0$, then

$$X^{',3}_{(a_{12,18,19},0)} := \{x_{1,10,11}(\epsilon a_{12,18,19}/a_{21}a_{22}a_{26}) \mid \epsilon \in \mathbb{F}_2\}$$

and

$$Y^{',3}_{(a_{12,18,19},0)} := \{x_{3,13}(\omega_{21,22}s)x_8(\epsilon a_{21}a_{22}a_{26}/c_{12,18,19}^2)x_9(c_{12,18,19}^2a_{20}s^2) \mid s \in \mathbb{F}_q, \epsilon \in \mathbb{F}_2\},$$

with $V^3_{(a_{12,18,19},0)} = X^{',3}_{(a_{12,18,19},0)} Y^{',3}_{(a_{12,18,19},0)} Z/(\ker\lambda)$ abelian. We get the family $\mathcal{F}_4^{2,p=2}$ in Table 5.1. The case $c_{12,18,19} = 0$ and $a_{14,15,17} := c_{14,15,17} \neq 0$ is symmetric, namely

$$X^{',3}_{(0,a_{14,15,17})} := \{x_{5,6,7}(\epsilon a_{14,15,17}/a_{20}a_{21}a_{22}) \mid \epsilon \in \mathbb{F}_2\}$$

and

$$Y^{',3}_{(0,a_{14,15,17})} := \{x_{3,13}(\omega_{21,22}s)x_8(c^2_{14,15,17}a_{26}s^2)x_9(\epsilon a_{20}a_{21}a_{22}/c^2_{14,15,17}) \mid s \in \mathbb{F}_q, \epsilon \in \mathbb{F}_2\},$$

and $V^3_{(0,a_{14,15,17})} = X^{',3}_{(0,a_{14,15,17})} Y^{',3}_{(0,a_{14,15,17})} Z/(\ker\lambda)$ is abelian, and we get the family $\mathcal{F}_4^{3,p=2}$ in Table 5.1.

Finally, if $a_{12,18,19} := c_{12,18,19} \neq 0$ and $a_{14,15,17} := c_{14,15,17} \neq 0$, we have that

$$X^{',3}_{(a_{12,18,19},a_{14,15,17})} := \{x_{1,10,11}(\epsilon a_{12,18,19}/(a_{21}a_{22}a_{26}))x_{5,6,7}(\epsilon a_{14,15,17}/(a_{20}a_{21}a_{22})) \mid \epsilon \in \mathbb{F}_2\}$$

and

$$Y^{',3}_{(a_{12,18,19},a_{14,15,17})} := \{x_{3,13}(\omega_{21,22}s + a_{12,18,19}a_{14,15,17}s^2)x_8(a^2_{14,15,17}a_{26}s^2)\cdot$$
$$\cdot\, x_9(\epsilon a_{20}a_{21}a_{22}/(a^2_{14,15,17}) + a^2_{12,18,19}a_{20}s^2) \mid s \in \mathbb{F}_q, \epsilon \in \mathbb{F}_2\},$$

and $V^3_{(a_{12,18,19},a_{14,15,17})}$ is abelian. We obtain the family $\mathcal{F}_4^{4,p=2}$ in Table 5.1.

The $[4, 24, 43]$-core in D_6**.** We have that

- $\mathcal{S} = \{\alpha_1, \dots, \alpha_{24}\}$,
- $\mathcal{Z} = \{\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}$,
- $\mathcal{A} = \mathcal{L} = \emptyset$,
- $\mathcal{I} = \{\alpha_1, \alpha_5, \alpha_6\}$ and $\mathcal{J} = \{\alpha_{17}, \alpha_{18}, \alpha_{19}\}$.

The form of Equation (4.3) is

$$s_{17}(a_{21}t_5 + a_{22}t_6) + s_{18}(-a_{21}t_1 - a_{23}t_6) + s_{19}(-a_{22}t_1 - a_{23}t_5) = 0.$$

Let $p \neq 2$. Then $X' = Y' = 1$, and $\overline{V} = X_2X_3X_4X_7 \cdots X_{16}X_{20}Z/(\ker\lambda)$. Notice that $X_2 \cap [\overline{V}, \overline{V}] = X_4 \cap [\overline{V}, \overline{V}] = 1$, and $[X_i, X_{20}] \neq 1$ just for $i = 2, 4$. Then we can take X_2X_4 for a candidate of an arm and X_{20} for a candidate of a leg. We have

$$[x_{20}(s_{20}), x_2(t_2)x_4(t_4)] = x_{23}(-s_{20}t_2)x_{24}(-s_{20}t_4).$$

Hence we apply the Reduction Lemma with $X^{',2} = X_{2,4} = \{x_{2,4}(t) \mid t \in \mathbb{F}_q\}$ and $Y^{',2} = 1$, reducing to $V^2 = X_{2,4}X_3X_7 \cdots X_{16}Z/(\ker\lambda)$; here, we have

$$X_{2,4} := \{x_{2,4}(t) \mid t \in \mathbb{F}_q\}, \quad x_{2,4}(t) := x_2(a_{24}t)x_4(-a_{23}t).$$

We have that $X_{12}X_{14}X_{15}$ is a subgroup of V^2, and that

eq:71011 (2.1) $[X_{12}X_{14}X_{15}, X_i] \neq 1 \Rightarrow i \in \{7, 10, 11\}$ and $X_i \cap [V^2, V^2] = 1$ for $i \in \{7, 10, 11\}$

We then apply the Reduction Lemma with $X_7X_{10}X_{11}$ and $X_{12}X_{14}X_{15}$ as candidates for an arm and a leg respectively, reducing to studying the equation

eq:tri2 (2.2) $$s_{12}(a_{21}t_{10} + a_{22}t_{11}) + s_{14}(-a_{21}t_7 - a_{24}t_{11}) + s_{15}(-a_{22}t_7 - a_{24}t_{10}) = 0.$$

As $p \neq 2$, we have that $X^{',3} = Y^{',3} = 1$. We reduce to $V^3 = X_{2,4}X_3X_8X_9X_{13}X_{16}Z/(\ker\lambda)$.

We observe that in V^3 we have that if $k = 8, 9$, then $[X_k, X_i] \neq 1$ just for $i = 16$, that $[X_{2,4}, X_i] = X_{13}$ if $i = 8, 9$, and that $X_{2,4} \cap [V^3, V^3] = 1 = X_{16} \cap [V^3, V^3]$. Moreover, we notice that X_{13} is central in V^3; we extend extend λ to $\lambda^{c_{13}}$ in the usual way for every $c_{13} \in \mathbb{F}_q$. If

$a_{13} := c_{13} \neq 0$, we apply the Reduction Lemma with $X_{2,4}X_{16}$ as a candidate for an arm, and X_8X_9 as a candidate for a leg. We have that

eq:2416 (2.3) $$\lambda([x_{2,4}(t)x_{16}(t_{16}), x_8(s_8)x_9(s_9)]) = \phi(a_{13}t(a_{24}s_9 + a_{23}s_8) + t_{16}(a_{24}s_9 - a_{23}s_8)).$$

We get that $X^{',4}_{(a_{13})} = Y^{',4}_{(a_{13})} = 1$, and $V^{',4}_{(a_{13})} = X_3X_{13}Z/(\ker\lambda)$ is abelian. We obtain the family $\mathcal{F}_5^{1,p\neq 2}$ in Table 5.1.

Let us now assume that $c_{13} = 0$. We examine $V^3_{(0)}$, and we notice that in this case $[X_{2,4}, X_i] = 1$ if $i = 8, 9$. Hence we apply the Reduction Lemma with leg X_8X_9, and smaller arm X_{16}. We get the expression as in Equation (2.3) by replacing a_{13} with 0. We obtain $X^{',4}_{(0)} = X_{8,9} := \{x_{8,9}(t) \mid t \in \mathbb{F}_q\}$ and $Y^{',4}_{(0)} = 1$. Here, we have $x_{8,9}(t) := x_8(a_{24}t)x_9(a_{23}t)$ for every $t \in \mathbb{F}_q$. Notice that $X_{8,9}$ is central in $V^{',4}_{(0)} = X_{2,4}X_3X_{8,9}Z/(\ker\lambda)$; we call by $\lambda'' = \lambda'^{c_{8,9}}$ the usual extension of λ' to $X_{8,9}$ for every $c_{8,9} \in \mathbb{F}_q$.

If $a_{8,9} := c_{8,9} \neq 0$, then we have

$$\lambda([x_{2,4}(s), x_3(t)]) = \lambda(x_{8,9}(st)) = \phi(a_{8,9}st).$$

The Reduction Lemma applies again, with arm $X_{2,4}$ and leg X_3, and we reduce to $V^5_{(0,a_{8,9})} = X_{8,9}Z/(\ker\lambda')$. We get the family $\mathcal{F}_5^{2,p\neq 2}$ in Table 5.1. Finally, if $a_{8,9} := c_{8,9} \neq 0$ then $V^5_{(0,0)} := X_{2,4}X_3Z/(\ker\lambda')$ is abelian; this gives the family $\mathcal{F}_5^{3,p\neq 2}$ in Table 5.1.

Let us now assume $p = 2$. In this case, we have

$$X' := \{x_{1,5,6}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{17,18,19}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{1,5,6}(t) := x_1(a_{23}t)x_5(a_{22}t)x_6(a_{21}t) \quad \text{and} \quad x_{17,18,19}(s) := x_{17}(a_{23}s)x_{18}(a_{22}s)x_{19}(a_{21}s),$$

and $\overline{V} = X_2X_3X_4X_7 \cdots X_{16}X_{20}X'Y'Z/(\ker\lambda)$. In a similar way to the case $p \neq 2$ after computing X' and Y', we notice that we can apply the Reduction Lemma with $X^{',2} = X_{2,4}$ and $Y^{',2} = 1$. We reduce to $V^2 = X_{2,4}X_3X_7 \cdots X_{16}X'Y'Z/(\ker\lambda)$. We notice that Y' is central in V^2; let us denote by $\lambda' := \lambda^{c_{17,18,19}}$ the usual extension of λ.

Suppose $a_{17,18,19} := c_{17,18,19} \neq 0$. In the group V, we have

$$[X_i, X_{13}] \neq 1 \Rightarrow i \in \{1,5,6\} \quad \text{and} \quad X_i \cap [V,V] = 1 \text{ for } i \in \{1,5,6\}.$$

We can then apply the Reduction Lemma with $X_{1,5,6}$ as a candidate for an arm, and X_{13} as a candidate for a leg. In V^2, we have

$$[x_{13}(s_{13}), x_{1,5,6}(t)] = x_{17,18,19}(s_{13}t)x_{21}(a_{22}a_{23}s_{13}t)x_{22}(a_{21}a_{23}s_{13}t)x_{23}(a_{21}a_{22}s_{13}t),$$

hence applying λ' we obtain the following equation,

eq:15613 (2.4) $$\phi(a_{17,18,19}s_{13}t + a_{21}a_{22}a_{23}s_{13}t^2) = 1.$$

We have that

$$X^{',3}_{(a_{17,18,19})} = \{1, x_{1,5,6}(a_{17,18,19}/(a_{21}a_{22}a_{23}))\} \text{ and } Y^{',3}_{(a_{17,18,19})} := \{1, x_{13}(a_{21}a_{22}a_{23}/(c^2_{17,18,19}))\},$$

and $V^3_{(a_{17,18,19})} = X_{2,4}X_3X_7 \cdots X_{12}X_{14}X_{15}X_{16}X^{',3}_{(a_{17,18,19})}Y^{',3}_{(a_{17,18,19})}Y'Z/(\ker\lambda')$. In this subquotient, we have that $[X_{2,4}, X_{12}X_{14}X_{15}] \cup X_{17,18,19} \neq 0$, that $X_{2,4} \cap [V^3_{(a_{17,18,19})}, V^3_{(a_{17,18,19})}] = 1$, and that Equation (2.1) holds. Moreover, recall that in V we have that if $k \in \{7, 10, 11\}$, then

$[X_i, X_j] \cap X_k \neq 1$ implies $i \in \{2,4\}$ or $j \in \{2,4\}$. We can then take $X_{2,4}X_7X_{10}X_{11}$ and $X_{12}X_{14}X_{15}$ as candidates for an arm and a leg respectively. We get the equation

$$\lambda([x_{12}(s_{12})x_{14}(s_{14})x_{15}(s_{15}), x_{2,4}(t)x_7(t_7)x_{10}(t_{10})x_{11}(t_{11})]) = \lambda(x_{17}(a_{23}s_{12}t_1)x_{18}(a_{24}s_{14}t_1)\cdot$$

$$\cdot x_{19}(a_{24}s_{15}t_1))\phi(s_{12}(a_{21}t_{10} + a_{22}t_{11}) + s_{14}(a_{21}t_7 + a_{24}t_{11}) + s_{15}(a_{22}t_7 + a_{24}t_{10})) = 1.$$

We get that $X^{',4}_{(a_{17,18,19})} = X_{7,10,11} := \{x_{7,10,11}(t) \mid t \in \mathbb{F}_q\}$ and $Y^{',4}_{(a_{17,18,19})} = 1$, where for every $t \in \mathbb{F}_q$ we have $x_{7,10,11}(t) := x_7(a_{24}t)x_{10}(a_{22}t)x_{11}(a_{21}t)$, and

$$V^4_{(a_{17,18,19})} = X_3X_{7,10,11}X_8X_9X_{16}X^{',3}_{(a_{17,18,19})}Y^{',3}_{(a_{17,18,19})}Y'Z/(\ker\lambda').$$

Notice that $X^{',4}_{(a_{17,18,19})}X_{7,10,11}$ is a subgroup of $V_{a_{17,18,19}}$, and that X_3 is there a direct product factor. Observe then that $[X_8, X_9] = [X_{16}, X_{7,10,11}] = 1$, and that

$$\lambda([x_8(s_8)x_9(s_9), x_{7,10,11}(t)x_{16}(t_{16})]) = \lambda(x_{17}(a_{24}s_9t)x_{18}(a_{22}s_8t)x_{19}(a_{21}s_8t))\phi(a_{23}s_8t_{16}+a_{24}s_9t_{16}).$$

As $a_{17,18,19} \neq 0$, applying the Reduction Lemma with arm $X_{7,10,11}X_{16}$ and leg $X_{7,10,11}X_{16}$ yields $X^{',5}_{(a_{17,18,19})} = Y^{',5}_{(a_{17,18,19})} = 1$ and the subquotient $V^5_{(a_{17,18,19})} = X_3X^{',3}_{(a_{17,18,19})}Y^{',3}_{(a_{17,18,19})}Y'Z/(\ker\lambda')$ of V, which is abelian. We obtain the family $\mathcal{F}_5^{1,p=2}$ in Table 5.1.

Let us now assume $c_{17,18,19} = 0$. As done for $c_{17,18,19} \neq 0$, we take $X_{1,5,6}$ and X_{13} as candidates for arm and leg respectively, but as we have no $a_{17,18,19}$ term in Equation (2.4) we now get $X^{',3}_{(0)} = Y^{',3}_{(0)} = 1$ and $V^3_{(0)} = X_{2,4}X_3X_7\cdots X_{12}X_{14}X_{15}X_{16}Z/(\ker\lambda')$. Notice that in this subquotient we have $[X_{2,4}, X_j] = 1$ for $j = 8, 9, 16$, and that $[X_{16}, X_i] \neq 1$ implies $i \in \{8,9\}$. We can apply the Reduction Lemma with X_{16} as a candidate for an arm, and X_8X_9 as a candidate for a leg. We have that

$$\lambda([x_8(s_8)x_9(s_9), x_{16}(t_{16})]) = \lambda(x_{23}(s_8t_{16})x_{24}(s_9t_{16})) = \phi(t_{16}(a_{23}s_8 + a_{24}s_9)).$$

We then get $X^{',4}_{(0)} = 1$ and $Y^{',4}_{(0)} = \{x_{8,9}(s) \mid s \in \mathbb{F}_q\}$, where $x_{8,9}(s) = x_8(a_{24}s)x_9(a_{23}s)$ for every $s \in \mathbb{F}_q$, and

$$V^4_{(0)} = X_{2,4}X_3X_7X_{8,9}X_{10}X_{11}X_{12}X_{14}X_{15}Z/(\ker\lambda').$$

Now we observe that (2.1) holds with $V^4_{(0)}$ in place of V^2, as X_{20} and $X_{17}X_{18}X_{19}$ are trivial in $V^4_{(0)}$. We take $X_7X_{10}X_{11}$ as a candidate for an arm and $X_{12}X_{14}X_{15}$ as a candidate for a leg. Equation (2.2) yields in this case

$$X^{',5}_{(0)} := X_{7,10,11} = \{x_{7,10,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y^{',5}_{(0)} := X_{12,14,15} = \{x_{12,14,15}(s) \mid s \in \mathbb{F}_q\},$$

and $V^5_{(0)} = X_{2,4}X_3X_{8,9}X_{7,10,11}X_{12,14,15}Z/(\ker\lambda')$. Here, for $s, t \in \mathbb{F}_q$ we have

$$x_{7,10,11}(t) = x_7(a_{24}t)x_{10}(a_{22}t)x_{11}(a_{21}t) \quad \text{and} \quad x_{12,14,15}(s) = x_{12}(a_{24}s)x_{14}(a_{22}s)x_{15}(a_{21}s).$$

Finally, we observe that $X_{8,9}$ and $X_{12,14,15}$ are central in $V^5_{(0)}$; we extend λ' to $\lambda'' := \lambda'^{c_{8,9},c_{12,14,15}}$ in the usual way. Observe that $[X_{2,4}, X_{7,10,11}] = 1$. We can then take $X_{2,4}X_{7,10,11}$ and X_3 as candidates for an arm and a leg respectively. We study

$$\lambda([x_3(s_3), x_{2,4}(t_1)x_{7,10,11}(t_2), x_2(s_2)x_3(s_3)x_5(s_5)]) = \phi(s_3(c_{8,9}t_1 + c_{12,14,15}t_2 + a_{21}a_{22}a_{24}t_2^2)) = 1.$$

If $a_{8,9} := c_{8,9} \neq 0$ and $b_{12,14,15} := c_{12,14,15}$ is arbitrary in $\mathbb{F}_q$, we have that

$X^{',6}_{(0,a_{8,9},b_{12,14,15})} = \{x_{2,4}((b_{12,14,15}t + a_{21}a_{22}a_{24}t^2)/(a_{8,9}^2))x_{7,10,11}(t) \mid t \in \mathbb{F}_q\}$ and $Y^{',6}_{(0,a_{8,9},b_{12,14,15})} = 1$, and $V^6_{(0,a_{8,9},b_{12,14,15})} = X^{',6}_{(0,a_{8,9},b_{12,14,15})} X_{8,9} Y^{',5}_{(0)} Z/(\ker \lambda'')$ is abelian; this gives the family $\mathcal{F}_5^{2,p=2}$ in Table 5.1. If $c_{8,9} = 0$ and $a_{12,14,15} := c_{12,14,15} \neq 0$, we have that

$$X^{',6}_{(0,0,a_{12,14,15})} = X_{2,4}\{1, x_{7,10,11}(c_{12,14,15}/(a_{21}a_{22}a_{24}))\} \text{ and } Y^{',6}_{(0,0,a_{12,14,15})} = \{1, x_3(a_{21}a_{22}a_{24}/(c^2_{12,14,15}))\},$$

and $V^6_{(0,0,a_{12,14,15})} = X^{',6}_{(0,0,a_{12,14,15})} Y^{',6}_{(0,0,a_{12,14,15})} Y^{',5}_{(0)} Z/(\ker \lambda'')$ is abelian; we obtain the family $\mathcal{F}_5^{4,p=2}$ in Table 5.1. If $c_{8,9} = c_{12,14,15} = 0$, we have that

$$X^{',6}_{(0,0,0)} = X_{2,4} \quad \text{and} \quad Y^{',6}_{(0,0,a_{12,14,15})} = 1,$$

and $V^6_{(0,0,0)} = X_{2,4} Z/(\ker \lambda'')$ is abelian. This yields the family $\mathcal{F}_5^{3,p=2}$ in Table 5.1.

The $[5, 18, 18]$-core in D_6. Before examining this core in detail, we recall the following, which is useful in the examination of this core. Let us consider the equation

eq:quad1 (2.5)
$$(at_1 + bt_2)^2 + ct_1 + dt_2 = 0$$

with indeterminates t_1, t_2 over the field $\mathbb{F}_q$ with $q = 2^f$.

- If $ad \neq bc$, then the solutions of Equation (2.5) are $t_1 = g_1(t) = g_1^{a,b,c,d}(t)$ and $t_2 = g_2(t) = g_2^{a,b,c,d}(t)$ for $t \in \mathbb{F}_q$, where
$$g_1(t) = \frac{t(d + bt)}{ad + bc} \quad \text{and} \quad g_2(t) = \frac{t(c + at)}{ad + bc}.$$
- If $ad = bc$, then the solutions of Equation (2.5) are $t_1 = h_1(t) = h_1^{a,b,c,d}(t)$ and $t_2 = h_2(t, \epsilon) = h_2^{a,b,c,d}(t, \epsilon)$ for $t \in \mathbb{F}_q$ and $\epsilon \in \mathbb{F}_2$, where
$$h_1(t) = bt \quad \text{and} \quad h_2(t, \epsilon) = at + \epsilon d/(b^2).$$
Hence there are $2q$ solutions if $c \neq 0$ and $d \neq 0$, and q solutions if $c = d = 0$.

We now proceed with the core analysis. We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{24}, \alpha_{25}\}$,
- $\mathcal{Z} = \{\alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{24}, \alpha_{25}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_5, \alpha_6\}$ and $\mathcal{L} = \{\alpha_{21}, \alpha_{22}, \alpha_{23}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{16}\}$ and $\mathcal{J} = \{\alpha_8, \alpha_9, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{20}\}$.

The form of Equation (4.3) is

$$s_8(a_{18}t_{10} + a_{19}t_{11}) + s_9(-a_{17}t_7 - a_{24}t_{16}) + s_{12}(a_{17}t_4 + a_{25}t_{16}) + s_{14}(-a_{18}t_2 - a_{24}t_{11}) + s_{15}(-a_{19}t_2 - a_{24}t_{10}) + s_{20}(-a_{24}t_4 - a_{25}t_7) = 0.$$

If $p \neq 2$. Then $X' = Y' = 1$, and $\overline{V} = X_3 Z/(\ker \lambda)$ is abelian. We get the family $\mathcal{F}_6^{p\neq 2}$ in Table 5.1.

Let then $p = 2$. Then we have that $X' := X_1' X_2'$ and $Y_1' Y_2'$, where

$$X_1' := \{x_{2,10,11}(t_1) \mid t_1 \in \mathbb{F}_q\} \quad \text{and} \quad X_1' := \{x_{4,7,16}(t_2) \mid t_2 \in \mathbb{F}_q\},$$
$$Y_1' := \{x_{8,14,15}(s_1) \mid s_1 \in \mathbb{F}_q\} \quad \text{and} \quad Y_2' := \{x_{9,12,20}(s_2) \mid s_2 \in \mathbb{F}_q\}$$

and for every $s_1, s_2, t_1, t_2 \in \mathbb{F}_q$,

$$x_{2,10,11}(t_1) := x_2(a_{24}t_1)x_{10}(a_{19}t_1)x_{11}(a_{18}t_1) \quad \text{and} \quad x_{4,7,16}(t_2) := x_4(a_{25}t_2)x_7(a_{24}t_2)x_{16}(a_{17}t_2),$$
$$x_{8,14,15}(s_1) := x_8(a_{24}s_1)x_{14}(a_{19}s_1)x_{15}(a_{18}s_1) \quad \text{and} \quad x_{9,12,20}(s_2) := x_9(a_{25}s_2)x_{12}(a_{24}s_2)x_{20}(a_{17}s_2).$$

We easily check that X' is a subgroup of $\overline{V} = X_3X'Y'Z/(\ker\lambda)$. We extend in the usual way λ to $\lambda' = \lambda^{c_{8,14,15},c_{9,12,20}}$ for every $c_{8,14,15}, c_{9,12,20} \in \mathbb{F}_q$. We get in $\overline{V}$

$$[x_3(s_3), x_{2,10,11}(t_1)x_{1,7,16}(t_2)] = x_{8,14,15}(s_3t_1)x_{9,12,20}(s_3t_2)x_{17}(a_{24}a_{25}s_3t_2^2)x_{18}(a_{19}a_{24}s_3t_1^2)\cdot$$
$$\cdot x_{19}(a_{18}a_{24}s_3t_1^2)x_{24}(a_{17}a_{25}s_3t_2^2 + a_{18}a_{19}s_3t_1^2)x_{25}(a_{17}a_{24}s_3t_2^2).$$

We take X' and X_3 as candidates for arm and leg respectively. We apply λ to the above equality, and we use Remark 5.1 and the remarks at the beginning of the examination to study the equation

$$\lambda([x_3(s_3), x_{2,10,11}(t_1)x_{4,7,16}(t_2)]) = \phi(s_3t_1(c_{8,14,15}+a_{18}a_{19}a_{24}t_1)+s_3t_2(c_{9,12,20}+a_{17}a_{24}a_{25}t_2)) = 1.$$

We denote by $\omega_{18,19,24}$ (respectively $\omega_{17,24,25}$) the unique square root of $a_{18}a_{19}a_{24}$ (respectively $a_{17}a_{24}a_{25}$). If $c^*_{8,14,15} := c_{8,14,15} \neq b_{9,12,20}\omega_{18,19,24}/\omega_{17,24,25}$ and $b_{9,12,20} := c_{9,12,20} \in \mathbb{F}_q$, then $X'^{,2}_{(c^*_{8,14,15},b_{9,12,20})} = \{x_{2,4,7,10,11,16}(t) \mid t \in \mathbb{F}_q\}$ and $Y'^{,2}_{(c^*_{8,14,15},b_{9,12,20})} = 1$, where $x_{2,4,7,10,11,16}(t) := x_{2,10,11}(g_1(t))x_{4,7,16}(g_2(t))$ for every $t \in \mathbb{F}_q$, and g_1, g_2 are as defined before. We have that $V^2_{(c^*_{8,14,15},b_{9,12,20})}$ is abelian, and we get the family $\mathcal{F}_6^{1,p=2}$ in Table 5.1.

Let us suppose that $c_{8,14,15} = c_{9,12,20}\omega_{18,19,24}/\omega_{17,24,25}$ and $c^*_{9,12,20} := c_{9,12,20} \neq 0$. Then

$$X'^{,2}_{(c_{8,14,15},c^*_{9,12,20})} := \{x_{2,4,7,10,11,16}(t,\epsilon) \mid t \in \mathbb{F}_q, \epsilon \in \mathbb{F}_2\}$$

and

$$Y'^{,2}_{(c_{8,14,15},c^*_{9,12,20})} := \{1, x_3(a_{18}a_{19}a_{24}/(c^2_{8,14,15})),$$

where $x_{2,4,7,10,11,16}(t,\epsilon) := x_{2,10,11}(h_1(t))x_{4,7,16}(h_2(t,\epsilon))$ for every $t \in \mathbb{F}_q$ and $\epsilon \in \mathbb{F}_2$, and h_1, h_2 are as previously defined. We have $V^2_{(c_{8,14,15},c^*_{9,12,20})}$ abelian. Hence we get the family $\mathcal{F}_6^{3,p=2}$ in Table 5.1. Finally, if $c_{8,14,15} = c_{9,12,20} = 0$, we have

$$X'^{,2}_{(0,0)} := \{x_{2,4,7,10,11,16}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y'^{,2}_{(0,0)} = 1,$$

where $x_{2,4,7,10,11,16}(t) := x_{2,10,11}(h_1(t))x_{4,7,16}(h_2(t,0))$, and $V^2_{(0,0)}$ is abelian. This gives the family $\mathcal{F}_6^{2,p=2}$ in Table 5.1

The $[6,19,20]$**-core in** D_6**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{24}, \alpha_{25}\}$,
- $\mathcal{Z} = \{\alpha_{13}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{24}, \alpha_{25}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_5, \alpha_6\}$ and $\mathcal{L} = \{\alpha_{21}, \alpha_{22}, \alpha_{23}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{16}\}$ and $\mathcal{J} = \{\alpha_8, \alpha_9, \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{20}\}$.

The form of Equation (4.3) is

$$s_8(a_{13}t_4 + a_{18}t_{10} + a_{19}t_{11}) + s_9(-a_{13}t_2 - a_{17}t_7 - a_{24}t_{16}) + s_{12}(a_{17}t_4 + a_{25}t_{16})$$
$$+ s_{14}(-a_{18}t_2 - a_{24}t_{11}) + s_{15}(-a_{19}t_2 - a_{24}t_{10}) + s_{20}(-a_{24}t_4 - a_{25}t_7) = 0.$$

If $p = 2$, then $X' = Y' = 1$, and $\overline{V} = X_3Z/(\ker\lambda)$ is abelian. We obtain the family $\mathcal{F}_7^{p=2}$ in Table 5.1.

We then let $p \neq 2$. If $a^*_{25} := a_{25} \neq 4a_{17}a_{18}a_{19}/(a^2_{13})$, then $X' = Y' = 1$, and we obtain the family $\mathcal{F}_7^{1,p\neq 2}$ in Table 5.1. Let now $a_{25} = 4a_{17}a_{18}a_{19}/(a^2_{13}))$. Then we have

$$X := \{x_{2,4,7,10,11,16}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y := \{x_{8,9,12,14,15,20}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{2,4,7,10,11,16}(t) := x_2(2a_{17}a_{24}t)x_4(a_{13}a_{25}t)x_7(-a_{13}a_{24}t)x_{10}(-2a_{17}a_{19}t)x_{11}(-2a_{17}a_{18}t)x_{16}(-a_{13}a_{17}t),$$

$$x_{8,9,12,14,15,20}(s) := x_8(2a_{17}a_{24}s)x_9(-a_{13}a_{25}s)x_{12}(-a_{13}a_{24}s)x_{14}(2a_{17}a_{19}s)x_{15}(2a_{17}a_{18}s)x_{20}(a_{13}a_{17}s).$$

We consider the usual extension of λ to $\lambda' = \lambda^{c_{8,9,12,14,15,20}}$ for every $c_{8,9,12,14,15,20} \in \mathbb{F}_q$. We have that

$$\lambda([x_{2,4,7,10,11,16}(t), x_3(s_3)]) = \lambda(x_{8,9,12,14,15,20}(s_3t)) = \phi(c_{8,9,12,14,15,20}s_3t).$$

If $a_{8,9,12,14,15,20} := c_{8,9,12,14,15,20} \neq 0$, then we apply the Reduction Lemma taking X' and X_3 as candidates for an arm and a leg respectively. We have $X'^{,2}_{(a_{8,9,12,14,15,20})} = Y'^{,2}_{(a_{8,9,12,14,15,20})} = 1$ and V^2 is abelian. We get the family $\mathcal{F}_7^{2,p\neq 2}$ in Table 5.1. Finally notice that if $c_{8,9,12,14,15,20} = 0$ then $V_{(0)} = X'X_3Z/(\ker\lambda)$ is already abelian; this gives the family $\mathcal{F}_7^{3,p\neq 2}$ in Table 5.1.

The $[3, 10, 9]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_4, \alpha_7, \alpha_{11}, \alpha_{12}, \alpha_{16}, \alpha_{19}, \alpha_{23}, \alpha_{24}, \alpha_{29}, \alpha_{31}\}$,
- $\mathcal{Z} = \{\alpha_{23}, \alpha_{29}, \alpha_{31}\}$,
- $\mathcal{A} = \{\alpha_3, \alpha_5, \alpha_9, \alpha_{10}, \alpha_{15}, \alpha_{18}, \alpha_{21}\}$ and $\mathcal{L} = \{\alpha_6, \alpha_{14}, \alpha_{20}, \alpha_{22}, \alpha_{25}, \alpha_{26}, \alpha_{28}\}$,
- $\mathcal{I} = \{\alpha_7, \alpha_{11}, \alpha_{19}\}$ and $\mathcal{J} = \{\alpha_{12}, \alpha_{16}, \alpha_{24}\}$.

The form of Equation (4.3) is

$$s_{12}(a_{23}t_{11} - a_{29}t_{19}) + s_{16}(-a_{23}t_7 + a_{31}t_{19}) + s_{24}(a_{29}t_7 + a_{31}t_{11}) = 0.$$

As remarked in Example 3.4, this quattern group is isomorphic to the quattern group corresponding to $\{\alpha_1, \ldots, \alpha_{10}\} \setminus \{\alpha_3\}$ in type D_4, which is in turn isomorphic to the quattern group arising from the previously examined $[3, 10, 9]$-core in D_6. The study proceeds in the same way; we refer to the families $\mathcal{F}_2^{p\neq 2}$ and $\mathcal{F}_2^{1,p=2}$, $\mathcal{F}_2^{2,p=2}$ in Table 5.2.

The $[5, 16, 15]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{21}\}$,
- $\mathcal{Z} = \{\alpha_{15}, \alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{21}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_6\}$ and $\mathcal{L} = \{\alpha_{13}, \alpha_{14}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_{11}\}$ and $\mathcal{J} = \{\alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{16}\}$.

The form of Equation (4.3) is

$$s_8(-a_{17}t_7 + a_{20}t_{11}) + s_9(a_{15}t_5 + a_{21}t_{11}) + s_{10}(-a_{15}t_3 - a_{18}t_7) + \\ s_{12}(-a_{17}t_2 + a_{18}t_5) + s_{16}(-a_{20}t_2 - a_{21}t_3) = 0.$$

If $p \neq 2$, then $X' = Y' = 1$ and $\overline{V}$ is abelian. We get the family $\mathcal{F}_7^{p\neq 2}$ in Table 5.2.

Let now $p = 2$. Then we have

$$X' := \{x_{2,3,5,7,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{8,9,10,12,16}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{2,3,5,7,11}(t) = x_2(a_{18}a_{21}t)x_3(a_{18}a_{20}t)x_5(a_{17}a_{21}t)x_7(a_{15}a_{20}t)x_{11}(a_{15}a_{17}t)$$

and

$$x_{8,9,10,12,16}(s) := x_8(a_{18}a_{21}s)x_9(a_{18}a_{20}s)x_{10}(a_{17}a_{21}s)x_{12}(a_{15}a_{20}s)x_{16}(a_{15}a_{17}s),$$

and $\overline{V} = X_4X'Y'Z/(\ker\lambda)$. We denote by $\lambda' = \lambda^{c_{8,9,10,12,16}}$ for every $c_{8,9,10,12,16} \in \mathbb{F}_q$ the usual extension of λ. In $\overline{V}$, we have that

$$\lambda([x_4(s), x_{2,3,5,7,11}(t)]) = \phi(c_{8,9,10,12,16}st + a_{15}a_{17}a_{18}a_{20}a_{21}st^2),$$

hence we take X' and X_4 as candidates for arm and leg in $\overline{V}$ respectively. We apply Remark 5.1. If $c_{8,9,10,12,16}$, then $X'^{,2}_{(0)} = Y'^{,2}_{(0)} = 1$ and we get the family $\mathcal{F}_7^{1,p=2}$ in Table 5.2. If $a_{8,9,10,12,16} := c_{8,9,10,12,16} \neq 0$, we get

$$X'^{,2}_{(a_{8,9,10,12,16})} = \{1, x_{2,3,5,7,11}(a_{8,9,10,12,16}/(a_{15}a_{17}a_{18}a_{20}a_{21}))\}$$

and

$$Y'^{,2}_{(a_{8,9,10,12,16})}\{1, x_4(a_{15}a_{17}a_{18}a_{20}a_{21}/a_{8,9,10,12,16}^2)\},$$

and $V^2_{(a_{8,9,10,12,16})}$ is abelian in this case as well. This gives the family $\mathcal{F}_7^{2,p=2}$ in Table 5.2.

The $[5, 20, 25]$**-core in** E_6**.** We have that

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{21}, \alpha_{24}\}$,
- $\mathcal{Z} = \{\alpha_{17}, \alpha_{18}, \alpha_{20}, \alpha_{21}, \alpha_{24}\}$,
- $\mathcal{A} = \{\alpha_4\}$ and $\mathcal{L} = \{\alpha_{19}\}$,
- $\mathcal{I} = \{\alpha_1, \alpha_6, \alpha_8, \alpha_9, \alpha_{10}\}$ and $\mathcal{J} = \{\alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{14}, \alpha_{15}\}$.

The form of Equation (4.3) is

$$s_7(a_{17}t_8 + a_{18}t_{10}) + s_{11}(-a_{20}t_8 - a_{21}t_9) + s_{13}(-a_{17}t_1 + a_{24}t_{10}) + \\ s_{14}(a_{20}t_6 + a_{24}t_9) + s_{15}(-a_{18}t_1 + a_{21}t_6 + a_{24}t_8) = 0.$$

Let $p \neq 3$. Then $X' = Y' = 1$, and $\overline{V} = X_2X_3X_5X_{12}X_{16}Z/(\ker\lambda)$. Observe that in $\overline{V}$ two copies of root subgroups commutate nontrivially just in the following cases,

eq:com520 (2.6) $\quad [X_2, X_{12}] = X_{17}, \quad [X_2, X_{16}] = X_{20}, \quad [X_3, X_{16}] = X_{21}, \quad [X_5, X_{12}] = X_{18},$

and of course $X_i \cap [\overline{V}, \overline{V}] = 1$. We apply the Reduction Lemma with $X_2X_3X_5$ as a candidate for an arm, and $X_{12}X_{16}$ as a candidate for a leg. We have

$$[x_2(t_2)x_3(t_3)x_5(t_5), x_{12}(s_{12})]x_{16}(s_{16}) = x_{17}(s_{12}t_2)x_{18}(-s_{12}t_5)x_{20}(s_{16}t_2)x_{21}(s_{16}t_3),$$

hence

$$\lambda([x_2(t_2)x_3(t_3)x_5(t_5), x_{12}(s_{12})]) = \phi(s_{12}(a_{17}t_2 - a_{18}t_5) + s_{16}(a_{20}t_2 + a_{21}t_3)).$$

We get $X'^{,2} = \{x_{2,3,5}(t) \mid t \in \mathbb{F}_q\}$ and $Y'^{,2} = 1$, where

$$x_{2,3,5}(t) := x_2(a_{18}a_{21}t)x_3(-a_{18}a_{20}t)x_5(a_{17}a_{21}t)$$

for every $t \in \mathbb{F}_q$. As $V^2 = X''Z/(\ker\lambda)$ is abelian, we get the family $\mathcal{F}_8^{p\neq 3}$ in Table 5.2.

Let us now assume that $p = 3$. Then we have

$$X' := \{x_{1,6,8,9,10}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{7,11,13,14,15}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{1,6,8,9,10}(t) := x_1(a_{21}a_{24}t)x_6(-a_{18}a_{24}t)x_8(-a_{18}a_{21}t)x_9(a_{18}a_{20}t)x_{10}(a_{17}a_{21}t)$$

and

$$x_{7,11,13,14,15}(s) := x_7(a_{20}a_{24}s)x_{11}(-a_{17}a_{24}s)x_{13}(-a_{18}a_{20}s)x_{14}(-a_{17}a_{21}s)x_{15}(a_{17}a_{20}s),$$

and $\overline{V} = X_2X_3X_5X_{12}X_{16}X'Y'Z/(\ker\lambda)$. We extend λ to $\lambda' = \lambda^{c_{7,11,13,14,15}}$, $c_{7,11,13,14,15} \in \mathbb{F}_q$.

Notice that X' is a subgroup of $\overline{V}$. Moreover, the nontrivial commutator relations in $\overline{V}$ are as in (2.6), and $[X', X_i] \neq 1$ if and only if $i \in \{2, 3, 5\}$, in which case the commutator lies inside Y'. In this case, the Reduction Lemma applies with $X'X_{12}X_{16}$ as a candidate for an arm and $X_2X_3X_5$ as a candidate for a leg. We study the equation

$$\begin{aligned}
&\lambda([x_{1,6,8,9,10}(t)x_{12}(t_{12})x_{16}(t_{16}), x_2(s_2)x_3(s_3)x_5(s_5)]) = \lambda(x_7(-a_{21}a_{24}s_3t)x_{11}(-a_{18}a_{24}s_5t))\cdot\\
&\quad \cdot \lambda(x_{13}(a_{18}a_{20}s_2t - a_{18}a_{21}s_3t)x_{14}(a_{18}a_{21}s_5t + a_{17}a_{21}s_2t)x_{15}(-a_{18}a_{20}s_5t + a_{17}a_{21}s_3t))\cdot\\
&\quad \cdot \phi(s_2(a_{17}t_{12} + a_{20}t_{16} + a_{17}a_{18}a_{20}a_{21}a_{24}t^2) + s_3(a_{21}t_{16} - a_{17}a_{18}a_{21}^2a_{24}t^2))\cdot\\
&\quad \cdot \phi(s_5(-a_{18}t_{12} - a_{18}^2a_{20}a_{21}a_{24}t^2)) = 1.
\end{aligned}$$

If $a_{7,11,13,14,15} := c_{7,11,13,14,15} \neq 0$, then we have that $X'^{,2}_{(a_{7,11,13,14,15})} = Y'^{,2}_{(a_{7,11,13,14,15})} = 1$, and $V^2_{(a_{7,11,13,14,15})} = Y'Z/(\ker\lambda')$ is abelian. We get the family $\mathcal{F}_8^{1,p=3}$ in Table 5.2. If $c_{7,11,13,14,15} = 0$, then we have

$$X'^{,2}_{(0)} = \{x_{1,6,8,9,10}(t)x_{12}(a_{18}a_{20}a_{21}a_{24}t^2)x_{16}(a_{17}a_{18}a_{21}a_{24}t^2) \mid t \in \mathbb{F}_q\},$$

$$Y'^{,2}_{(0)} = \{x_2(a_{18}a_{21}s)x_3(-a_{18}a_{20}s)x_5(a_{17}a_{21}s) \mid s \in \mathbb{F}_q\},$$

and $V^2_{(0)} = X'^{,2}_{(0)}Y'^{,2}_{(0)}Z/(\ker\lambda')$ is abelian. This yields the family $\mathcal{F}_8^{2,p=3}$ in Table 5.2.

The $[5, 21, 30]$-core in E_6. We have that

- $\mathcal{S} = \{\alpha_1, \ldots, \alpha_{24}\}$,
- $\mathcal{Z} = \{\alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{21}\}$,
- $\mathcal{A} = \mathcal{L} = \emptyset$,
- $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ and $\mathcal{J} = \{\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}\}$.

As remarked before, this core has been studied in [**?**, Section 3]; we refer to it and to the families $\mathcal{F}_9^{p\neq 3}$ and $\mathcal{F}_9^{1,p=3}$, $\mathcal{F}_9^{2,p=3}$, $\mathcal{F}_9^{3,p=3}$ in Table 5.2.

The $[6, 15, 12]$-core in E_6. We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{18}, \alpha_{20}, \alpha_{22}, \alpha_{23}\}$,
- $\mathcal{Z} = \{\alpha_8, \alpha_9, \alpha_{15}, \alpha_{20}, \alpha_{22}, \alpha_{23}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_5, \alpha_{11}\}$ and $\mathcal{L} = \{\alpha_{12}, \alpha_{17}, \alpha_{21}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_6, \alpha_7\}$ and $\mathcal{J} = \{\alpha_{14}, \alpha_{16}, \alpha_{18}\}$.

The form of Equation (4.3) is

$$s_{14}(a_{20}t_6 - a_{22}t_7) + s_{16}(-a_{20}t_2 - a_{23}t_7) + s_{18}(-a_{22}t_2 + a_{23}t_6) = 0.$$

Let $p \neq 2$. Then $X' = Y' = 1$ and $\overline{V} = X_3X_4X_{10}Z/(\ker\lambda)$. Notice that $[X_4, X_{10}] = 1$, and

$$\lambda([x_3(s_3), x_4(t_4)x_{10}(t_{10})]) = \lambda(x_9(s_3t_4)x_{15}(s_3t_{10})) = \phi(s_3(a_9t_4 + a_{15}t_{10})).$$

We consider X_4X_{10} as a candidate for an arm, and X_3 as a candidate for a leg. We get $X'^{,2} := \{x_4(a_{15}t)x_{10}(-a_9t) \mid t \in \mathbb{F}_q\} := 1$ and $Y'^{,2} = 1$, and $\overline{V} = X'^{,2}Z/(\ker\lambda)$ is now abelian. This yields the family $\mathcal{F}_{13}^{p\neq 2}$ in Table 5.2.

We now assume $p = 2$. We have that

$$X' := \{x_{2,6,7}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{14,16,18}(s) \mid s \in \mathbb{F}_q\},$$

where for every $s, t \in \mathbb{F}_q$,

$$x_{2,6,7}(t) := x_2(a_{23}t)x_6(a_{22}t)x_7(-a_{20}t) \quad \text{and} \quad x_{14,16,18}(s) := x_{14}(a_{23}s)x_{16}(a_{22}s)x_{18}(-a_{20}s),$$

and $\overline{V} = X_3X_4X_{10}X'Y'Z/(\ker\lambda)$. We put $\lambda' = \lambda^{c_{14,16,18}}$ the usual extension of λ, for every $c_{14,16,18} \in \mathbb{F}_q$. Notice that in $\overline{V}$ we have analogous commutator relations for copies of root subgroups as in the case $p \neq 2$, and we have

$$1 \neq [X', X_4] \subseteq X_8, \qquad 1 \neq [X', X_{10}] \subseteq Y' \quad \text{and} \quad [X', X_3] = 1.$$

We apply the Reduction Lemma with X_3X' and X_4X_{10} as candidates for an arm and a leg respectively. We get

$$[x_3(t_3)x_{2,6,7}(t), x_4(s_4)x_{10}(s_{10})] = x_{14,16,18}(s_{10}t)x_8(a_{23}s_4t)x_9(s_4t_3)x_{15}(s_{10}t_3)\cdot$$
$$\cdot\, x_{20}(a_{22}a_{23}s_{10}t^2)x_{22}(a_{20}a_{23}s_{10}t^2)x_{23}(a_{20}a_{22}s_{10}t^2),$$

and we apply λ' to the above, in order to study the equation

$$\phi(s_{10}(c_{14,16,18}t + a_{15}t_3 + a_{20}a_{22}a_{23}t^2) + s_4(a_8a_{23}t + a_9t_3)) = 1.$$

Let us first assume that $c^*_{14,16,18} := c_{14,16,18} \neq a_8a_{15}a_{23}/a_9$. Let us put $b := a_9c_{14,16,18}+a_8a_{15}a_{23}$ and $a := a_9a_{20}a_{22}a_{23}$. Then we have that

$$X^{',2}_{(c^*_{14,16,18})} = \{1, x_{2,6,7}(b/a)x_3(a_8a_{23}b/(a_9a))\} \text{ and } Y^{',2}_{(c^*_{14,16,18})} = \{1, x_4(a_{15}a/b^2)x_{10}(a_9a/b^2)\},$$

and $V^2_{(c^*_{14,16,18})}$ is abelian. This gives the family $\mathcal{F}^{2,p=2}_{13}$ in Table 5.2.

If we now assume that $c_{14,16,18} = a_8a_{15}a_{23}/a_9$, then we have that $X^{',2}_{(c_{14,16,18})} = Y^{',2}_{(c_{14,16,18})} = 1$, and $V^2_{(c_{14,16,18})}$ is abelian. We obtain the family $\mathcal{F}^{1,p=2}_{13}$ in Table 5.2.

The $[6, 17, 17]$-core in E_6**.** We have that

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{20}, \alpha_{23}\}$,
- $\mathcal{Z} = \{\alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{17}, \alpha_{20}, \alpha_{23}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_6\}$ and $\mathcal{L} = \{\alpha_{18}, \alpha_{21}\}$,
- $\mathcal{I} = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_{11}\}$ and $\mathcal{J} = \{\alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{16}\}$.

The form of Equation (4.3) is

$$s_8(-a_{13}t_3 + a_{14}t_5 - a_{17}t_7 + a_{20}t_{11}) + s_9(-a_{13}t_2 + a_{15}t_5) + s_{10}(-a_{14}t_2 - a_{15}t_3)+$$
$$s_{12}(-a_{17}t_2 + a_{23}t_{11}) + s_{16}(-a_{20}t_2 - a_{23}t_7) = 0.$$

We first assume $p \neq 2$. If $a^*_{23} := a_{23} \neq -a_{15}a_{17}a_{20}/(a_{13}a_{14})$, then $X' = Y' = 1$ and $\overline{V} = X_4Z/(\ker\lambda)$ is abelian, which gives the family $\mathcal{F}^{1,p\neq2}_{15}$ in Table 5.2. Let us now assume $a_{23} = -a_{15}a_{17}a_{20}/(a_{13}a_{14})$. Then we have that

$$X' := \{x_{2,3,5,7,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{8,9,10,12,16}(s) \mid s \in \mathbb{F}_q\},$$

where for every s and t in $\mathbb{F}_q$,

$$x_{2,3,5,7,11}(t) := x_2(a_{15}a_{17}a_{20}t)x_3(-a_{14}a_{17}a_{20}t)x_5(a_{13}a_{17}a_{20}t)x_7(a_{13}a_{14}a_{20}t)x_{11}(-a_{13}a_{14}a_{17}t),$$

$$x_{8,9,10,12,16}(s) := x_8(a_{15}a_{17}a_{20}s)x_9(-a_{14}a_{17}a_{20}s)x_{10}(-a_{13}a_{17}a_{20}s)x_{12}(a_{13}a_{14}a_{20}s)x_{16}(a_{13}a_{14}a_{17}s),$$

and $\overline{V} = X_4X'Y'Z/(\ker\lambda)$. We extend λ to $\lambda' = \lambda^{c_{8,9,10,12,16}}$ in the usual way for every $c_{8,9,10,12,16} \in \mathbb{F}_q$. In $\overline{V}$, we have that

$$[x_{2,3,5,7,11}(t), x_4(s)] = x_{8,9,10,12,16}(st)x_{13}(a_{14}a_{15}a_{17}^2a_{20}^2st^2)x_{14}(a_{13}a_{15}a_{17}^2a_{20}^2st^2)\cdot$$
$$\cdot x_{15}(a_{13}a_{14}a_{17}^2a_{20}^2st^2)x_{17}(a_{13}a_{14}a_{15}a_{17}a_{20}^2st^2)x_{20}(a_{13}a_{14}a_{15}a_{17}^2a_{20}st^2)x_{23}(a_{13}^2a_{14}^2a_{17}a_{20}st^2).$$

Let us first suppose $a_{8,9,10,12,16} := c_{8,9,10,12,16} \neq 0$. Observe that X' and X_4 can be taken, respectively, as a candidate for an arm and a candidate for a leg in $\overline{V}$. Applying λ' to the above and replacing a_{23} with $-a_{15}a_{17}a_{20}/(a_{13}a_{14})$, we study the equation

$$\lambda([x_{2,3,5,7,11}(t), x_4(s_4)]) = \phi(c_{8,9,10,12,16}st) = 1.$$

We have that $X'^{,2}_{(a_{8,9,10,12,16})} = Y'^{,2}_{(a_{8,9,10,12,16})} = 1$, and $V^2_{(a_{8,9,10,12,16})} = Y'Z/(\ker\lambda')$ is abelian. This gives the family $\mathcal{F}_{15}^{2,p\neq 2}$ in Table 5.2. If $c_{8,9,10,12,16} = 0$, then we notice that $V^2_{(0)} = X_4X'Z/(\ker\lambda')$ is abelian; this yields the family $\mathcal{F}_{15}^{3,p\neq 2}$ in Table 5.2.

Let us now assume $p = 2$. We have that

$$X' := \{x_{2,3,5,7,11}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' := \{x_{8,9,10,12,16}(s) \mid s \in \mathbb{F}_q\},$$

but this time $x_{2,3,5,7,11}(t)$ and $x_{8,9,10,12,16}(s)$ are substantially different from the case $p \neq 2$, namely for $s, t \in \mathbb{F}_q$,

$$x_{2,3,5,7,11}(t) := x_2(a_{15}a_{23}t)x_3(a_{14}a_{23}t)x_5(a_{13}a_{23}t)x_7(a_{15}a_{20}t)x_{11}(-a_{15}a_{17}t),$$
$$x_{8,9,10,12,16}(s) := x_8(a_{15}a_{23}s)x_9(a_{14}a_{23}s)x_{10}(a_{13}a_{23}s)x_{12}(a_{15}a_{20}s)x_{16}(-a_{15}a_{17}s),$$

and $\overline{V} = X_4X'Y'Z/(\ker\lambda)$. We define $\lambda' = \lambda^{c_{8,9,10,12,16}}$ as before.

We now proceed as in the case $p \neq 2$. In $\overline{V}$, we have that

$$[x_{2,3,5,7,11}(t), x_4(s)] = x_{8,9,10,12,16}(st)x_{13}(a_{14}a_{15}a_{23}^2st^2)x_{14}(a_{13}a_{15}a_{23}^2st^2)\cdot$$
$$\cdot x_{15}(a_{13}a_{14}a_{23}^2st^2)x_{17}(a_{15}^2a_{20}a_{23}st^2)x_{20}(a_{15}^2a_{17}a_{23}st^2)x_{23}(a_{15}^2a_{17}a_{20}st^2).$$

Applying λ', we study the equation

$$\phi(c_{8,9,10,12,16}st + a_{15}a_{23}(a_{13}a_{14}a_{23} + a_{15}a_{17}a_{20})t^2) = 1.$$

If $a_{8,9,10,12,16} := c_{8,9,10,12,16} \neq 0$ and $a_{23}^* := a_{23} \neq a_{15}a_{17}a_{20}/a_{13}a_{14}$, then defining $b := a_{8,9,10,12,16}$ and $a := a_{15}a_{23}(a_{13}a_{14}a_{23} + a_{15}a_{17}a_{20})$ we get

$$X'^{,2}_{(a_{8,9,10,12,16})} = \{1, x_{2,3,5,7,11}(b/a)\} \quad \text{and} \quad Y'^{,2}_{(a_{8,9,10,12,16})} = \{1, x_4(a/(b^2))\},$$

and $V^2_{(a_{8,9,10,12,16})} = X'^{,2}_{(a_{8,9,10,12,16})}Y'^{,2}_{(a_{8,9,10,12,16})}Z/(\ker\lambda')$ is abelian. We get the family $\mathcal{F}_{15}^{3,p=2}$ in Table 5.2. If $a_{8,9,10,12,16} := c_{8,9,10,12,16} \neq 0$ and $a_{23} = a_{15}a_{17}a_{20}/a_{13}a_{14}$, or if $c_{8,9,10,12,16} = 0$ and $a_{23}^* := a_{23} \neq a_{15}a_{17}a_{20}/a_{13}a_{14}$, then the Reduction Lemma applies with arm X' and leg Y', and we reduce to the abelian subquotient $Y'Z/(\ker\lambda')$. We obtain, respectively, the families $\mathcal{F}_{15}^{1,p=2}$ and $\mathcal{F}_{15}^{2,p=2}$ in Table 5.2. Finally, if $c_{8,9,10,12,16} = 0$ and $a_{23} = a_{15}a_{17}a_{20}/a_{13}a_{14}$, we have that $V^2_{(0)} = X'X_4Z/(\ker\lambda')$ is abelian, and this gives the family $\mathcal{F}_{15}^{4,p=2}$ in Table 5.2.