Contents

We collect here the calculations for the determination of the structure constants of the endomorphism algebra of a Gelfand-Graev representation in the case $G = \mathrm{SO}_5(q)$. Throughout, we assume that p is an odd prime for G.

1 Type B2: calculation of structure constants

1.1 333

[] (of type) :

$$1 = 1$$

$$\sum \psi(1) = 1$$

Should equal (1.1)

1.2 311

[] (of type) :

$$1 = t_2(c_2/c_3) \Leftrightarrow c_2 = c_3$$

$$\sum \psi(1) = 1$$

Should equal (1.5)

1.3 322

[] (of type) :

$$1 = t_1(d_2/d_3) \Leftrightarrow d_2 = d_3$$

$$\sum \psi(1) = 1$$

Should equal (1.16)

1.4 300

[] (of type) :

$$1 = t_1(a_2/a_3)t_2(b_2/b_3) \Leftrightarrow a_2 = a_3,\ b_2 = b_3$$

$$\sum \psi(1) = 1$$

Should equal (1.27)

1.5 131

$[2, 1, 2]$ (of type CCC) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, :

$$t_2(c_1) = t_2(c_3) \Leftrightarrow c_1 = c_3$$

$$\sum \psi(1) = 1$$

Should equal (1.2)

1.6 113

$[2,1,2]$ (of type AAA) $x_1 \in k$, $x_2 \in k$, $x_3 \in k$, :

$$t_2(c_1) = t_2(c_2) \Leftrightarrow c_1 = c_2$$

$$\sum \psi(1) = q^3$$

Should equal (1.6)

1.7 111

$[2,0,2]$ (of type CBA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, :

$$t_2(c_1) = t_1(-1)t_2(c_2c_3x_2^2) \Leftrightarrow x_2^2 = \frac{c_1}{c_2c_3},\ 1 = -1$$

The intersection is empty, so the constant is 0!

$$\sum \psi(u_1(-x_3/x_2))$$

Should equal (1.7)

1.8 112

$[2,1,0]$ (of type ACB) $x_1 \in k$, $x_2 = 1$, $x_3 \in k^*$, :

$$t_2(c_1) = t_1(1/(d_3x_3))t_2(c_2d_3^2x_3^2) \Leftrightarrow x_3 = \frac{1}{d_3},\ x_3^2 = \frac{c_1}{c_2d_3^2},\ \frac{c_2}{d_3^2} = \frac{c_1}{d_3^2},\ c_1 = c_2$$

$$\sum \psi(u_2(-1/(c_2x_3))) = q\phi(-\frac{d_3}{c_2})$$

Should equal (1.8)

1.9 110

$[0,1,0]$ (of type BCB) $x_1 \in k^*$, $x_2 = 1$, $x_3 \in k^*$, :

$$t_2(c_1) = t_1(x_1/(a_3x_3))t_2(a_3^2b_3c_2x_3^2) \Leftrightarrow x_3 = \frac{x_1}{a_3},\ x_1^2 = \frac{c_1}{c_2b_3}.$$

Hence

$$\sum \psi(u_2(-x_1 - \frac{1}{c_2x_3} - \frac{1}{b_3x_1})) = \sum_{\zeta^2 = \frac{c_1}{b_3c_2}} \phi\left(\zeta + \frac{a_3}{c_2\zeta} + \frac{1}{b_3\zeta}\right) = \mathcal{S}_2(\frac{c_1}{b_3c_2}, 1, \frac{a_3}{c_2} + \frac{1}{b_3}).$$

Should equal (1.9)

1.10 121

$[0,1,2]$ (of type BAC) $x_1 \in k^*$, $x_2 \in k$, $x_3 = 1$, :

$$t_2(c_1) = t_1(-d_2x_1)t_2(c_3) \Leftrightarrow x_1 = -\frac{1}{d_2},\ c_1 = c_3$$

$$\sum \psi(u_1(x_2 + \frac{x_2}{d_2x_1})u_2(-\frac{1}{c_3x_1})) = q\phi(\frac{d_2}{c_3})$$

Should equal (1.17)

1.11 122

$[2,0,2]$ (of type ABC) $x_1 \in k$, $x_2 \in k^*$, $x_3 = 1$, :

$$t_2(c_1) = t_1(-d_2/d_3)t_2(d_3^2x_2^2) \Leftrightarrow d_2 = -d_3,\ x_2^2 = \frac{c_1}{d_3^2}$$

$$\sum \psi(u_1(x_2)) = q \sum_{\zeta^2 = c_1} \phi\left(\frac{\zeta}{d_3}\right).$$

Should equal (1.18)

1.12 120

$[0,0,2]$ (of type BBC) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 = 1$, :

$$t_2(c_1) = t_1(-d_2x_1/a_3)t_2(a_3^2b_3x_2^2) \Leftrightarrow x_1 = -\frac{a_3}{d_2},\ x_2^2 = \frac{c_1}{a_3^2b_3}$$

$$\sum \psi(u_1(-x_2 + \frac{1}{d_2x_1x_2})u_2(-x_1 - \frac{1}{b_3x_1})) = \sum \psi(u_1(-x_2 - \frac{1}{a_3x_2})u_2(\frac{a_3}{d_2} + \frac{d_2}{b_3a_3}))$$
$$= \sum_{\zeta^2 = \frac{b_3}{c_1}} \phi(\zeta + \frac{1}{a_3\zeta} + \frac{a_3}{d_2} + \frac{d_2}{b_3a_3}) = \phi(\zeta + \frac{1}{a_3\zeta} + \frac{a_3}{d_2} + \frac{d_2}{b_3a_3}) + \phi(-\zeta - \frac{1}{a_3\zeta}\frac{a_3}{d_2} + \frac{d_2}{b_3a_3})$$

where $\zeta^2 = \frac{b_3}{c_1}$. Should equal (1.19)

1.13 101

$[0,1,0]$ (of type BAB) $x_1 \in k^*$, $x_2 \in k$, $x_3 \in k^*$, :

$$t_2(c_1) = t_1(a_2x_1/x_3)t_2(b_2c_3x_3^2) \Leftrightarrow x_1 = \frac{x_3}{a_2},\ x_3^2 = \frac{c_1}{b_2c_3}$$

$$\sum \psi(u_1(x_2 - x_2x_3/(a_2x_1))u_2(-x_3 - 1/(c_3x_1) - 1/(b_2x_3))) = \sum_{\zeta^2 = c_1c_3b_2} \phi(\frac{c_1 + a_2b_2 + c_3}{\zeta})$$

Should equal (1.28)

1.14 102

$[2, 0, 0]$ (of type ABB) $x_1 \in k$, $x_2 \in k^*$, $x_3 \in k^*$, :

$$t_2(c_1) = t_1(a_2/(d_3x_3))t_2(b_2d_3^2x_2^2x_3^2) \Leftrightarrow x_3 = \frac{a_2}{d_3},\ x_2^2 = \frac{c_1}{b_2a_2^2}$$

$$\sum \psi(u_1(x_2 + 1/(a_2x_2))u_2(-x_3 - 1/(b_2x_3))) = \sum_{\zeta^2 = \frac{b_2}{c_1}} \phi(\zeta + \frac{1}{a_2\zeta} - \frac{d_3}{a_2b_2} - \frac{a_2}{d_3})$$

Should equal (1.29)

1.15 100

$[0, 0, 0]$ (of type BBB) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 \in k^*$, :

$$t_2(c_1) = t_1(a_2x_1/(a_3x_3))t_2(a_3^2b_2b_3x_2^2x_3^2) \Leftrightarrow x_1 = \frac{a_3}{a_2}x_3,\ x_2^2 = \frac{c_1}{a_3^2b_2b_3}\frac{1}{x_3^2}$$

$$\sum \psi(u_1(-x_2 - \frac{1}{a_2x_2} - \frac{x_3}{a_2x_1x_2})u_2(-x_1 - x_3 - \frac{1}{b_3x_1} - \frac{1}{b_2x_3}))$$

$$= \sum_{x_3 \in k^*} \sum_{\zeta^2 = \frac{c_1}{b_2b_3}} \phi(-\frac{\zeta}{a_3}\frac{1}{x_3} - \left[\frac{a_3}{a_2\zeta} + \frac{1}{\zeta}\right]x_3 - \left[\frac{a_3}{a_2} + 1\right]x_3 - \left[\frac{a_2}{a_3b_3} + \frac{1}{b_2}\right]\frac{1}{x_3})$$

$[2, 0, 2]$ (of type CBA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, :

$$t_2(c_1) = t_1(-a_2/a_3)t_2(a_3^2b_2b_3x_2^2) \Leftrightarrow a_2 = -a_3,\ x_2^2 = \frac{c_1}{a_3^2b_2b_3}$$

$$\sum \psi(u_1(-x_3/(a_2x_2))u_2(x_3)) = \sum_{\zeta^2 = \frac{c_1}{b_2b_3}} \sum_{x_3 \in k} \phi(\frac{x_3}{\zeta} + x_3) = 0$$

where $\zeta^2 = \frac{c_1}{b_2b_3}$. Should equal (1.30)

1.16 232

$[1, 2, 1]$ (of type CCC) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, :

$$t_1(d_1) = t_1(d_3) \Leftrightarrow d_1 = d_3$$

$$\sum \psi(1) = 1$$

Should equal (1.3)

1.17 211

$[1, 0, 1]$ (of type ABC) $x_1 \in k$, $x_2 \in k^*$, $x_3 = 1$, :

$$t_1(d_1) = t_1(-c_3x_2)t_2(c_2/c_3) \Leftrightarrow x_2 = -\frac{d_1}{c_3},\, c_2 = c_3$$

$$\sum \psi(u_2(-x_2)) = q\phi(\frac{d_1}{c_3})$$

Should equal (1.10)

1.18 212

$[0, 2, 1]$ (of type BAC) $x_1 \in k^*$, $x_2 \in k$, $x_3 = 1$, :

$$t_1(d_1) = t_1(-d_3)t_2(c_2x_1^2) \Leftrightarrow d_1 = -d_3,\, x_1^2 = \frac{1}{c_2}$$

$$\sum \psi(u_1(-1/(d_3x_1))u_2(x_2 - x_2/(c_2x_1^2))) = \phi(\frac{\zeta}{d_3}) + \phi(-\frac{\zeta}{d_3})$$

where $\zeta^2 = c_1$. Should equal (1.11)

1.19 210

$[0, 0, 1]$ (of type BBC) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 = 1$, :

$$t_1(d_1) = t_1(-a_3b_3x_2)t_2(c_2x_1^2/b_3) \Leftrightarrow x_2 = -\frac{d_1}{a_3b_3},\, x_1^2 = \frac{b_3}{c_2}$$

$$\sum \psi(u_1(-x_1 - \frac{1}{a_3x_1})u_2(-x_2 - \frac{1}{c_2x_1^2x_2})) = \sum \psi(u_1(-x_1 - \frac{1}{a_3x_1})u_2(\frac{d_1}{a_3b_3} + \frac{a_3}{d_1}))$$
$$= \sum_{\zeta^2 = \frac{b_3}{c_2}} \phi(\zeta + \frac{1}{a_3\zeta} + \frac{d_1}{a_3b_3} + \frac{a_3}{d_1})$$

Should equal (1.12)

1.20 223

$[1, 2, 1]$ (of type AAA) $x_1 \in k$, $x_2 \in k$, $x_3 \in k$, :

$$t_1(d_1) = t_1(-d_2) \Leftrightarrow d_1 = -d_2$$

$$\sum \psi(1) = q^3$$

Should equal (1.20)

1.21 221

$[1, 2, 0]$ (of type ACB) $x_1 \in k$, $x_2 = 1$, $x_3 \in k^*$, :

$$t_1(d_1) = t_1(c_3 d_2 x_3^2) t_2(1/(c_3 x_3^2)) \Leftrightarrow x_3^2 = \frac{d_1}{d_2 c_3} = \frac{1}{c_3}.$$

Hence we have $d_1 = d_2$, and

$$\sum \psi(u_1(-1/(d_2 x_3))) = q \sum_{\zeta^2 = \frac{1}{c_3}} \phi\left(\frac{1}{d_2 \zeta}\right).$$

Should equal (1.21)

1.22 222

$[1, 0, 1]$ (of type CBA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, :

$$t_1(d_1) = t_1(-d_2 d_3 x_2) \Leftrightarrow x_2 = -\frac{d_1}{d_2 d_3}$$

$$\sum \psi(u_2(-x_3^2/x_2)) = \sum_{x_3 \in k} \phi(\frac{d_2 d_3}{d_1} x_3^2) = \begin{cases} G & \text{if } \frac{d_2 d_3}{d_1} \text{ is a square in } \mathbb{F}_q, \\ -G & \text{otherwise.} \end{cases}$$

Should equal (1.22)

1.23 220

$[0, 2, 0]$ (of type BCB) $x_1 \in k^*$, $x_2 = 1$, $x_3 \in k^*$, :

$$t_1(d_1) = t_1(a_3 b_3 d_2 x_3^2) t_2(x_1^2/(b_3 x_3^2)) \Leftrightarrow x_3^2 = \frac{d_1}{a_3 b_3 d_2},\ x_1^2 = b_3 x_3^2 = \frac{d_1}{a_3 d_2}$$

$$\sum \psi(u_1(-x_1 - \frac{1}{d_2 x_3} - \frac{1}{a_3 x_1}) u_2(2\frac{x_3}{x_1}) = \sum_{\zeta_1^2 = \frac{d_1}{a_3 d_2}} \sum_{\zeta_3^2 = \frac{d_1}{a_3 b_3 d_2}} \phi\left(-\zeta_1 - \frac{1}{d_2 \zeta_3} - \frac{1}{a_3 \zeta_1} + 2\frac{\zeta_1}{\zeta_3}\right)$$

Should equal (1.23) - true, since interchanging d_1 and d_2 we interchange ζ_3 and $\frac{1}{a_3 \zeta_3}$.

1.24 201

$[1, 0, 0]$ (of type ABB) $x_1 \in k$, $x_2 \in k^*$, $x_3 \in k^*$, :

$$t_1(d_1) = t_1(-a_2 c_3 x_2 x_3^2) t_2(b_2/(c_3 x_3^2)) \Leftrightarrow x_3^2 = \frac{b_2}{c_3},\ x_2 = -\frac{d_1}{a_2 b_2}$$

$$\sum \psi(u_1(-x_3 - \frac{1}{a_2 x_3}) u_2(-x_2 - \frac{1}{b_2 x_2})) = \sum_{\zeta^2 = \frac{b_2}{c_3}} \sum_{x_1 \in k} \phi(-\zeta - \frac{1}{a_2 x_3} + \frac{d_1}{a_2 b_2} + \frac{a_2}{d_1})$$

$$= q\left[\sum_{\zeta^2=\frac{b_2}{c_3}} \phi(\zeta + \frac{1}{a_2\zeta} + \frac{d_1}{a_2b_2} + \frac{a_2}{d_1})\right]$$

where $\zeta^2 = \frac{b_2}{c_3}$. Should equal (1.31)

1.25 202

$[0, 2, 0]$ (of type BAB) $x_1 \in k^*$, $x_2 \in k$, $x_3 \in k^*$, :

$$t_1(d_1) = t_1(-a_2d_3x_3^2)t_2(b_2x_1^2/x_3^2) \Leftrightarrow x_3^2 = -\frac{d_1}{a_2d_3},\ x_3^2 = b_2x_1^2$$

$$\sum \psi(u_1(x_3 - \frac{1}{d_3x_1} + \frac{1}{a_2x_3})u_2(x_2 + 2\frac{x_3}{b_2x_1} - \frac{x_2x_3^2}{b_2x_1^2})) = \sum_{x_2\in k} \phi(x_3 - \frac{1}{d_3x_1} + \frac{1}{a_2x_3} + 2\frac{x_3}{b_2x_1})$$

$$q\sum_{\zeta_1^2=b_2} \phi(\zeta_1x_1 - \frac{1}{d_3x_1} + \frac{1}{a_2\zeta_1x_1} + 2\frac{1}{\zeta_1}) = q\sum_{\zeta_1^2=b_2, \zeta_2^2=-\frac{d_1}{a_2d_3b_2}} \phi(\zeta_1x_1 - \frac{1}{d_3\zeta_2} + \frac{1}{a_2\zeta_1\zeta_2} + 2\frac{1}{\zeta_1})$$

Should equal (1.32)

1.26 200

$[0, 0, 0]$ (of type BBB) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 \in k^*$, :

$$t_1(d_1) = t_1(-a_2a_3b_3x_2x_3^2)t_2(b_2x_1^2/(b_3x_3^2)) \Leftrightarrow x_2 = -\frac{d_1}{a_2a_3b_2x_1^2}, x_3^2 = \frac{b_2}{b_3}x_1^2,$$

with sum

$$\sum \psi(u_1(-x_1 - x_3 - 1/(a_3x_1) - 1/(a_2x_3))u_2(-(b_2x_1^2x_2^2 + x_1(x_1 + 2x_3) + x_3^2)/(b_2x_1^2x_2)))$$

$$= \sum \phi(-x_1 - x_3 - 1/(a_3x_1) - 1/(a_2x_3) - (b_2x_1^2x_2^2 + x_1(x_1 + 2x_3) + x_3^2)/(b_2x_1^2x_2))$$

$$= \sum \phi(-x_1 - x_3 - \frac{1}{a_3x_1} - \frac{1}{a_2x_3} - x_2 - \frac{1}{b_2x_2} - 2\frac{x_3}{b_2x_1x_2} - \frac{x_3^2}{b_2x_1^2x_2})$$

$$= \sum \phi(-x_1 - x_3 - \frac{1}{a_3x_1} - \frac{1}{a_2x_3} - x_2 - \frac{1}{b_2x_2} - 2\frac{x_3}{b_2x_1x_2} - \frac{1}{b_3x_2})$$

$$= \sum \phi(-x_1 - x_3 - x_2 - \frac{1}{a_3x_1} - \frac{1}{a_2x_3} - \frac{1}{b_2x_2} - \frac{1}{b_3x_2} - 2\frac{x_3}{b_2x_1x_2})$$

If $\frac{b_2}{b_3}$ is not a square in $\mathbb{F}_q$, then the sum is zero. Otherwise, we have that $x_2 = -\frac{d_1}{a_2a_3b_2x_1^2}$ and $x_3 = \pm\beta x_1$, where β is a square root of $\frac{b_2}{b_3}$. The sum is

$$\sum_{\epsilon\in\{-1,1\}}\sum_{x_1\in k^\times} \phi\left(-x_1 - \epsilon\beta x_1 + \frac{d_1}{a_2a_3b_2x_1^2} - \frac{1}{a_3x_1} - \epsilon\frac{1}{a_2\beta x_1} + \frac{1}{b_2}\frac{a_2a_3b_2x_1^2}{d_1} + \frac{1}{b_3}\frac{a_2a_3b_2x_1^2}{d_1} + \epsilon\frac{2}{b_2}1x_1\frac{a_2a_3b_2x_1^2}{d_1}\beta x_1\right)$$

$$= \sum_{\epsilon\in\{-1,1\}} \sum_{x_1\in k^\times} \phi\left(x_1^2(\frac{a_2a_3}{d_1} + \frac{a_2a_3b_2}{b_3d_1} + 2\epsilon\frac{a_2a_3\beta}{d_1}) + x_1(-1-\epsilon\beta) + \frac{1}{x_1}(-\frac{1}{a_3} - \epsilon\frac{1}{a_2\beta}) + \frac{1}{x_1^2}(\frac{d_1}{a_2a_3b_2})\right).$$

[1, 0, 1] (of type CBA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, :

$$t_1(d_1) = t_1(-a_2a_3b_3x_2)t_2(b_2/b_3) \Leftrightarrow x_2 = \frac{-d_1}{a_2a_3b_3},\ b_2 = b_3$$

$$\sum \psi(u_1(x_3)u_2(-x_3^2/(b_2x_2))) = \sum_{x_3\in k} \phi(x_3 + \frac{a_2a_3}{d_1}x_3^2) = \begin{cases} G\phi\left(-\frac{d_1}{4a_2a_3}\right) & \text{if } \frac{a_2a_3}{d_1} \text{ is a square in } \mathbb{F}_q, \\ -G\phi\left(-\frac{d_1}{4a_2a_3}\right) & \text{otherwise.} \end{cases}$$

Should equal (1.33)

1.27 030

[1, 2, 1, 2] (of type CCCC) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(a_3)t_2(b_3) \Leftrightarrow a_1 = a_3,\ b_1 = b_3$$

$$\sum \psi(1) = 1$$

Should equal (1.4)

1.28 011

[1, 0, 1, 0] (of type ABCB) $x_1 \in k$, $x_2 \in k^*$, $x_3 = 1$, $x_4 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(x_2/x_4)t_2(c_2c_3x_4^2) \Leftrightarrow x_2 = a_1x_4, \Leftrightarrow x_4^2 = \frac{b_1}{c_2c_3}$$

$$\sum \psi(u_2(-x_2 - \frac{1}{c_3x_4} - \frac{1}{c_2x_4})) = q\mathcal{S}_2(b_2c_2c_3, 0, a_1b_1 + c_2 + c_3).$$

Should equal (1.13)

1.29 012

[0, 2, 1, 0] (of type BACB) $x_1 \in k^*$, $x_2 \in k$, $x_3 = 1$, $x_4 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(d_3x_1^2/x_4)t_2(c_2x_4^2/x_1^2) \Leftrightarrow x_4 = \frac{d_3x_1^2}{a_1},\ x_1^2 = \frac{a_1^2b_1}{c_2d_3^2},\ x_4 = \frac{a_1b_1}{c_2d_3}$$

Hence

$$\sum \psi(u_1(-\frac{d_3x_4+1}{d_3x_1})u_2(-\frac{c_2x_1^2+1}{c_2x_4})) = q\sum_{\zeta^2=\frac{b_1}{c_2}} \phi(\zeta + \frac{1}{a_1\zeta} - \frac{d_3}{a_1b_1} - \frac{a_1}{d_3}).$$

Should equal (1.14)

1.30 010

$[0,0,1,0]$ (of type BBCB) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 = 1$, $x_4 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(a_3x_1^2x_2/x_4)t_2(b_3c_2x_4^2/x_1^2) \Leftrightarrow x_1^2 = \frac{a_1x_4}{a_3x_2},\ x_1^2 = \frac{b_3c_2}{b_1}x_4^2,\ x_2 = \frac{a_1b_1}{a_3b_3c_2}\frac{1}{x_4}$$

hence

$$\sum \psi(u_1(-x_1 - \frac{x_4}{x_1x_2} - \frac{1}{a_3x_1})u_2(-x_2 - \frac{1}{c_2x_4} - \frac{x_1^2}{b_3x_4} - \frac{1}{b_3x_2}))$$
$$= \sum_{\zeta^2 = \frac{c_2}{b_1b_3}} \mathcal{S}_{q-1}(1, -\zeta b_3 - \frac{a_3b_3}{a_1}\zeta - \frac{c_2}{b_1} - \frac{a_3c_2}{a_1b_1}, -\frac{b_1}{a_3c_2}\zeta - \frac{a_1b_1}{a_3b_3c_2} - \frac{1}{c_2}).$$

Should equal (1.15)

1.31 021

$[1,0,0,2]$ (of type ABBC) $x_1 \in k$, $x_2 \in k^*$, $x_3 \in k^*$, $x_4 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(-d_2x_2)t_2(c_3x_3^2) \Leftrightarrow x_2 = -\frac{a_1}{d_2},\ x_3^2 = \frac{b_1}{c_3}$$

$$\sum \psi(u_1(-x_3 + \frac{1}{d_2x_2x_3})u_2(-x_2 - \frac{1}{c_3x_2x_3^2})) = \sum_{x_1\in k} \psi(u_1(-x_3 - \frac{1}{a_1x_3})u_2(\frac{a_1}{d_2} + \frac{d_2}{b_1a_1}))$$
$$= q\left[\sum_{\zeta^2=\frac{b_1}{c_3}} \phi(\zeta + \frac{1}{a_1\zeta} + \frac{a_1}{d_2} + \frac{d_2}{b_1a_1})\right]$$

where $\zeta^2 = \frac{b_1}{c_3}$. Should equal (1.24)

1.32 022

$[0,2,0,2]$ (of type BABC) $x_1 \in k^*$, $x_2 \in k$, $x_3 \in k^*$, $x_4 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(-d_2d_3x_1^2)t_2(x_3^2/x_1^2) \Leftrightarrow x_1^2 = -\frac{a_1}{d_2d_3},\ x_1^2 = \frac{x_3^2}{b_1}$$

Hence

$$\sum \psi(u_1(x_3 - \frac{1}{d_3x_1} + \frac{1}{d_2x_1})u_2(2\frac{x_1}{x_3})) = q \sum_{\zeta_1^2=-\frac{a_1}{d_2d_3}} \sum_{\zeta_3^2=-\frac{a_1b_1}{d_2d_3}} \phi\left(\zeta_3 - \frac{1}{d_3\zeta_1} + \frac{1}{d_2\zeta_1} + 2\frac{\zeta_1}{\zeta_3}\right).$$

Should equal (1.25)

1.33 020

$[0,0,0,2]$ (of type BBBC) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 \in k^*$, $x_4 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(-a_3d_2x_1^2x_2)t_2(b_3x_3^2/x_1^2) \Leftrightarrow x_2 = -\frac{a_1}{a_3d_2x_1^2}, x_3^2 = \frac{b_1}{b_3}x_1^2,$$

with sum

$$\sum \psi(u_1(-x_1 - x_3 + 1/(d_2x_2x_3) + 1/(d_2x_1x_2) - 1/(a_3x_1))u_2(-(b_3x_2^2x_3^2 + x_1(x_1 + 2x_3) + x_3^2)/(b_3x_2x_3^2)))$$

$$\sum \phi(-x_1 - x_3 + \frac{1}{d_2x_2x_3} + \frac{1}{d_2x_1x_2} - \frac{1}{a_3x_1} - (b_3x_2^2x_3^2 + x_1(x_1 + 2x_3) + x_3^2)/(b_3x_2x_3^2))$$

$$\sum \phi(-x_1 - x_3 + \frac{1}{d_2x_2x_3} + \frac{1}{d_2x_1x_2} - \frac{1}{a_3x_1} - x_2 - \frac{x_1^2}{b_3x_2x_3^2} - 2\frac{x_1}{b_3x_2x_3} - \frac{1}{b_3x_2})$$

$$\sum \phi(-x_1 - x_3 + \frac{1}{d_2x_2x_3} + \frac{1}{d_2x_1x_2} - \frac{1}{a_3x_1} - x_2 - \frac{1}{b_1x_2} - 2\frac{x_1}{b_3x_2x_3} - \frac{1}{b_3x_2})$$

$$\sum \phi(-x_1 - x_3 - x_2 + \frac{1}{d_2x_2x_3} + \frac{1}{d_2x_1x_2} - \frac{1}{a_3x_1} - \frac{1}{b_1x_2} - \frac{1}{b_3x_2} - 2\frac{x_1}{b_3x_2x_3})$$

$$\sum_{\zeta^2 = \frac{b_1}{b_3}} \sum_{x_1 \in k^\times} \phi\left(x_1^2(2\frac{a_3d_2}{a_1b_3\zeta} + \frac{a_3d_2}{a_1b_3} + \frac{a_3d_2}{a_1b_1}) + x_1(-1 - \zeta - \frac{a_3}{a_1\zeta} - \frac{a_3}{a_1}) + \frac{1}{x_1}(-\frac{1}{a_3}) + \frac{1}{x_1^2}(\frac{a_1}{a_3d_2})\right).$$

$[1,0,1,2]$ (of type CBAC) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, $x_4 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(-a_3d_2x_2)t_2(b_3) \Leftrightarrow x_2 = -\frac{a_1}{a_3d_2}, \; b_1 = b_3$$

$$\sum \psi(u_1(x_3 + x_3/(d_2x_2))u_2(-x_3^2/(b_3x_2))) = \sum \phi(x_3 + \frac{x_3}{d_2x_2} - \frac{x_3^2}{b_3x_2}) =$$

$$= \sum \phi(x_3(1 - \frac{a_3}{a_1}) + \frac{a_3d_2}{a_1b_3}x_3^2) = \begin{cases} G\phi(-\frac{a_1b_3}{4a_3d_2}(1 - \frac{a_3}{a_1})^2) & \text{if } \frac{a_3d_2}{a_1b_3} \text{ is a square in } \mathbb{F}_q, \\ -G\phi(-\frac{a_1b_3}{4a_3d_2}(1 - \frac{a_3}{a_1})^2) & \text{otherwise.} \end{cases}$$

Should equal (1.26)

1.34 003

$[1,2,1,2]$ (of type AAAA) $x_1 \in k$, $x_2 \in k$, $x_3 \in k$, $x_4 \in k$, :

$$t_1(a_1)t_2(b_1) = t_1(a_2)t_2(b_2) \Leftrightarrow a_1 = a_2, \; b_1 = b_2$$

$$\sum \psi(1) = q^4$$

Should equal (1.34)

1.35 001

$[1,0,0,0]$ (of type ABBB) $x_1 \in k$, $x_2 \in k^*$, $x_3 \in k^*$, $x_4 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(a_2x_2/x_4)t_2(b_2c_3x_3^2x_4^2)$$

Then we have $a_1 = a_2x_2/x_4$ and $b_1 = b_2c_3x_3^2x_4^2$, hence $x_2 = \frac{a_1}{a_2}x_4$ and $(x_3x_4)^2 = \frac{b_1}{b_2c_3}$. The sum is

$$q \sum_{x_2,x_3,x_4 \text{ as above}} \psi(u_1(-x_3-1/(a_2x_3)-x_4/(a_2x_2x_3))u_2(-x_2-x_4-1/(c_3x_3^2x_4)-1/(c_3x_2x_3^2)-1/(b_2x_4)))$$

$$= q \sum_{\zeta^2=\frac{b_1}{b_2c_3}} \mathcal{S}_{q-1}(1, -1-\frac{a_1}{a_2}-\frac{b_2}{b_1}-\frac{a_2b_2}{a_1b_1}-\frac{1}{a_2\zeta}-\frac{1}{a_1\zeta}, \zeta-\frac{1}{b_2}).$$

$[1,2,0,2]$ (of type ACBA) $x_1 \in k$, $x_2 = 1$, $x_3 \in k^*$, $x_4 \in k$, :

$$t_1(a_1)t_2(b_1) = t_1(-a_2)t_2(b_2c_3x_3^2)$$

Then we have $a_1 = -a_2$ and $b_1 = b_2c_3x_3^2$, that is $x_3^2 = \frac{a_1}{b_2c_3}$. The sum is

$$q \sum_{x_4 \in k} \psi(u_1(-x_4/(a_2x_3))u_2(x_4 - x_4/(c_3x_3^2))) = q^2 \sum_{\zeta^2=\frac{b_1}{b_2c_3}} \delta_{\frac{1}{a_2\zeta}, 1-\frac{b_1}{b_2}}.$$

Should equal (1.35).

1.36 002

$[0,2,0,0]$ (of type BABB) $x_1 \in k^*$, $x_2 \in k$, $x_3 \in k^*$, $x_4 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(a_2d_3x_1^2/x_4)t_2(b_2x_3^2x_4^2/x_1^2)$$

We have that $a_1 = a_2d_3x_1^2/x_4$ and $b_1 = b_2x_3^2x_4^2/x_1^2$, hence $x_4 = \frac{a_2d_3}{a_1}x_1^2$ and $(x_1x_3)^2 = \frac{a_1^2b_1}{a_2^2b_2d_3^2}$. The sum is

$$\sum \psi(u_1(x_3 - 1/(d_3x_1) + 1/(a_2x_3) - x_4/(a_2x_1))u_2(-x_1^2/(x_3^2x_4) + 2x_1/x_3 - x_4 - 1/(b_2x_4)))$$

$$= \sum \phi\left(x_3 - \frac{1}{d_3x_1} - \frac{1}{a_2x_3} - \frac{x_4}{a_2x_1} - \frac{x_1^2}{x_3^2x_4} + 2\frac{x_1}{x_3} - x_4 - \frac{1}{b_2x_4}\right)$$

$$= q \sum_{\zeta^2=\frac{b_1}{b_2}} \tilde{S}_{q-1}(1, \tau_1', \tau_2', \tau_3', \tau_4'),$$

with

$$\tau_1' = a_2d_3(\frac{2}{a_1\zeta} - \frac{b_2}{a_1b_1} - \frac{1}{a_1}), \qquad \tau_2' = d_3(\frac{1}{a_1\zeta} - \frac{1}{a_1}), \qquad \tau_3' = \frac{a_1\zeta}{a_2d_3} - \frac{1}{d_3}, \qquad \tau_4' = -\frac{a_1}{a_2b_2d_3}.$$

[1, 0, 1, 2] (of type CBAA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, $x_4 \in k$, :

$$t_1(a_1)t_2(b_1) = t_1(a_2d_3x_2)t_2(b_2)$$

We must have that $b_1 = b_2$ (otherwise the sum is zero) and $a_1 = a_2d_3x_2$, hence $x_2 = \frac{a_1}{a_2d_3}$. The sum is The sum is

$$\sum_{x_2,x_3,x_4 \text{ as above}} \psi(u_1(-x_3/(a_2x_2))u_2(-x_3^2/x_2))$$

$$= q\sum_{x_3\in k} \phi\left(-\frac{d_3}{a_1}x_3 - \frac{a_2d_3}{a_1}x_3^2.\right)$$

$$= \begin{cases} qG\phi(\frac{d_3}{4a_1a_2}) & \text{if } -\frac{a_2d_3}{a_1} \text{ is a square} \\ -qG\phi(\frac{d_3}{4a_1a_2}) & \text{otherwise .} \end{cases}$$

Should equal (1.36)

1.37 000

[0, 2, 0, 2] (of type BCBA) We have that

$$a_1 = -a_2a_3x_1^2 \quad \text{and} \quad b_1 = \frac{b_2b_3x_3^2}{x_1^2}. \tag{1}$$

Hence

$$\sum \psi\left(u_1\left(-x_1 - \frac{1}{a_3x_1} - \frac{x_4}{a_2x_3} + \frac{1}{a_2x_1}\right)u_2\left(x_4 - \frac{x_1^2x_4}{b_3x_3^2} + \frac{2x_1}{b_3x_3}\right)\right)$$

with summation over $(x_1, x_2, x_3, x_4) \in \mathbb{F}_q^\times \times \{1\} \times \mathbb{F}_q^\times \times \mathbb{F}_q$. This equals

$$\sum \phi\left(-x_1 - \frac{1}{a_3x_1} - \frac{x_4}{a_2x_3} + \frac{1}{a_2x_1} + x_4 - \frac{x_1^2x_4}{b_3x_3^2} + \frac{2x_1}{b_3x_3}\right).$$

If $-\frac{a_1}{a_2a_3} \notin \mathbb{F}_{q,2}^\times$ or $\frac{b_1}{b_2b_3} \notin \mathbb{F}_{q,2}^\times$, then the equations (1) do not have a common solution, and the sum is zero. We assume $-\frac{a_1}{a_2a_3}, \frac{b_1}{b_2b_3} \in \mathbb{F}_{q,2}^\times$. Then we have that

$$(x_1, x_3) \in \left\{(\zeta_1, \zeta_1\zeta_2) \mid \zeta_1^2 = -\frac{a_1}{a_2a_3}, \zeta_2^2 = \frac{b_1}{b_2b_3}\right\}.$$

Hence we can write the sum as

$$\sum_{\zeta_1^2=-\frac{a_1}{a_2a_3}}\left(\sum_{\zeta_2^2=\frac{b_1}{b_2b_3}}\sum_{x_4\in k}\phi(x_4(1-\frac{b_2}{b_1}+\frac{a_3b_2b_3\zeta_1}{a_1b_1}\zeta_2)-\zeta_1+\frac{a_2}{a_1}\zeta_1-\frac{a_3}{a_1}\zeta_1+2\frac{b_2}{b_1}\zeta_2)\right),$$

and by (iii) in §?? this can be written as

$$\sum_{\zeta_1^2=-\frac{a_1}{a_2a_3}} q\delta_{\frac{b_1}{b_2b_3},-\frac{a_1a_2(b_1-b_2)^2}{a_3b_2^2b_3^2}}\phi\left(-\zeta_1+\frac{a_2}{a_1}\zeta_1-\frac{a_3}{a_1}\zeta_1-2\frac{b_2}{b_1}(1-\frac{b_2}{b_1})\frac{a_1b_1}{a_3b_2b_3\zeta_1}\right).$$

This sum evaluates to zero if $b_1 = b_2$, and to

$$q\delta_{b_3,-\frac{(b_1-b_2)^2 a_1 a_2}{a_3 b_1 b_2}} \sum_{\zeta_1^2=-\frac{a_1}{a_2 a_3}} \phi\left(-\zeta_1 + \frac{a_2}{a_1}\zeta_1 - \frac{a_3}{a_1}\zeta_1 + 2\frac{a_3 b_2}{a_1(b_1-b_2)}\zeta_1\right)$$

otherwise.

[1, 0, 1, 0] (of type CBAB) We have that

$$a_1 = \frac{a_2 a_3 x_2}{x_4} \quad \text{and} \quad b_1 = b_2 b_3 x_4^2.$$

Hence

$$\begin{aligned}\sum \psi\left(u_1\left(x_3 - \frac{x_3 x_4}{a_2 x_2}\right) u_2\left(-x_4 - \frac{1}{b_3 x_4} - \frac{x_3^2}{b_3 x_2} - \frac{1}{b_2 x_4}\right)\right) \\ = \sum \phi\left(x_3 - \frac{x_3 x_4}{a_2 x_2} - x_4 - \frac{1}{b_3 x_4} - \frac{x_3^2}{b_3 x_2} - \frac{1}{b_2 x_4}\right),\end{aligned}$$

with summation over $(x_1, x_2, x_3, x_4) \in \{1\} \times \mathbb{F}_q^\times \times \mathbb{F}_q \times \mathbb{F}_q^\times$.

If $\frac{b_1}{b_2 b_3} \notin \mathbb{F}_{q,2}^\times$, then we have no possible value for x_4, and the sum is zero. We then assume $\frac{b_1}{b_2 b_3} \in \mathbb{F}_{q,2}^\times$. In this case, we have

$$(x_2, x_4) \in \left\{(\frac{a_1}{a_2 a_3}\zeta, \zeta) \mid \zeta^2 = \frac{b_1}{b_2 b_3}\right\},$$

and we can write the above sum as

$$\sum_{\zeta^2=\frac{b_1}{b_2 b_3}} \sum_{x_3\in k} \phi(x_3 - \frac{x_3}{a_2}\frac{\zeta a_2 a_3}{a_1 \zeta} - \zeta - \frac{1}{b_3 \zeta} - \frac{x_3^2}{b_3}\frac{a_2 a_3}{a_1 \zeta} - \frac{1}{b_2 \zeta}) = \sum_{\zeta^2=\frac{b_1}{b_2 b_3}} \sum_{x_3\in k} \phi\left(A(\zeta)x_3^2 + Bx_3 + C(\zeta)\right),$$

where $A(\zeta) = \frac{a_2 a_3}{a_1 b_3 \zeta}$, $B = 1 - \frac{a_3}{a_1}$ and $C(\zeta) = \zeta + \frac{1}{b_2 \zeta} + \frac{1}{b_3 \zeta}$. The above sum can be written by §**??** as

$$G \sum_{\zeta^2=\frac{b_1}{b_2 b_3}} (2\delta_{A(\zeta)\in\mathbb{F}_{q,2}^\times} - 1)\phi\left(C(\zeta) - \frac{B^2}{4A(\zeta)}\right).$$

[0, 0, 0, 0] (of type BBBB). In this case, we have that

$$\sum \psi(u_1(-x_1 - x_3 - \frac{1}{a_3 x_1} - \frac{1}{a_2 x_3} - \frac{x_4}{a_2 x_2 x_3} - \frac{x_4}{a_2 x_1 x_2}) u_2(-x_2 - x_4 - \frac{x_1^2}{b_3 x_3^2 x_4} - \frac{x_1^2}{b_3 x_2 x_3^2} - \frac{2x_1}{b_3 x_2 x_3} - \frac{1}{b_3 x_2} - \frac{1}{b_2 x_4}))$$

$$= \sum \phi(-x_1 - x_3 - \frac{1}{a_3 x_1} - \frac{1}{a_2 x_3} - \frac{x_4}{a_2 x_2 x_3} - \frac{x_4}{a_2 x_1 x_2} - x_2 - x_4 - \frac{x_1^2}{b_3 x_3^2 x_4} - \frac{x_1^2}{b_3 x_2 x_3^2} - \frac{2x_1}{b_3 x_2 x_3} - \frac{1}{b_3 x_2} - \frac{1}{b_2 x_4}),$$

and by §**??** we have $a_1 = a_2 a_3 x_1^2 x_2 / x_4$ and $b_1 = b_2 b_3 x_3^2 x_4^2 / x_1^2$. The first equation yields $x_4 = \frac{a_2 a_3 x_1^2 x_2}{a_1}$. Substituting this into the second equation yields $(x_1 x_2 x_3)^2 = (\frac{a_1}{a_2 a_3})^2 \frac{b_1}{b_2 b_3}$.

If $\frac{b_1}{b_2 b_3} \notin \mathbb{F}_{q,2}^{\times}$, then the equalities are not both satisfied at the same time, hence the sum is zero. Let us then assume $\frac{b_1}{b_2 b_3} \notin \mathbb{F}_{q,2}^{\times}$. Let us put $A := \frac{a_1}{a_2 a_3}$. Then we have that

$$(x_1, x_2, x_3, x_4) \in \left\{ (t, u, \pm\frac{A\zeta}{tu}, \frac{t^2 u}{A}) \mid t, u \in k^{\times}, \zeta^2 = \frac{b_1}{b_2 b_3} \right\}.$$

The above sum can now be written as

$$\sum_{\zeta^2 = B, t, u \in k^{\times}} \phi(-t - \frac{A\zeta}{tu} - \frac{1}{a_3 t} - \frac{tu}{a_2 A\zeta} - \frac{t^3 u}{a_2 A^2 \zeta} - \frac{t}{a_2 A} - u - \frac{t^2 u}{A} - \frac{t^2 u}{b_3 A\zeta^2} - \frac{t^4 u}{b_3 A^2 \zeta^2} - 2\frac{t^2}{b_3 A\zeta} - \frac{1}{b^3 u} - \frac{A}{b_2 t^2 u}),$$

which can be put in the following more compact form,

$$\sum_{t \in k^{\times}} \phi\left(-\frac{2t^2}{b_3 A\zeta} - t\frac{a_2 A + 1}{a_2 A} - \frac{1}{a_3 t}\right) \mathcal{S}_{q-1}\left(1, -\frac{t^4}{b_3 A^2 \zeta^2} - \frac{t^3}{a_2 A^2 \zeta} - t^2 \frac{b_3 \zeta^2 + 1}{b_3 A\zeta^2} - \frac{t}{a_2 A\zeta} - 1, -\frac{1}{b_3} - \frac{A\zeta}{t} - \frac{A}{b_2 t^2}\right).$$