We collect here the calculations for the determination of the structure constants of the endomorphism algebra of a Gelfand-Graev representation in the case $G = \mathrm{PGL}_3(q)$.

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1 Type A2: calculation of structure constants

1.1 333

[] (of type) :

$$1 = 1$$

$$\sum \psi(1) = 1$$

Should equal (1.1)

1.2 311

[] (of type) :

$$1 = t_1(\frac{c_2}{c_3}) \Leftrightarrow c_2 = c_3$$

$$\sum \psi(1) = 1$$

Should equal (1.5)

1.3 322

[] (of type) :

$$1 = t_2(\frac{d_2}{d_3}) \Leftrightarrow d_2 = d_3$$

$$\sum \psi(1) = 1$$

Should equal (1.13)

1.4 300

[] (of type) :

$$1 = t_1(\frac{b_2}{b_3})t_2(\frac{a_2}{a_3}) \Leftrightarrow b_2 = b_3,\ a_2 = a_3$$

$$\sum \psi(1) = 1$$

Should equal (1.21)

1.5 131

$[1, 2]$ (of type CC) $x_1 = 1,\ x_2 = 1,$:

$$t_2(c_1) = t_2(c_3) \Leftrightarrow c_1 = c_3$$

$$\sum \psi(1) = 1$$

Should equal (1.2)

1.6 112

$[1,2]$ (of type AC) $x_1 \in k$, $x_2 = 1$, :

$$t_2(c_1) = t_1(\frac{-c_2}{d_3})t_2(-d_3) \Leftrightarrow c_1 = c_2 = -d_3$$

$$\sum \psi(u_1(-x_1)u_2(x_1)) = \sum_{x_1 \in k} \phi(1) = q$$

Should equal (1.6) (symmetric in c_1 and c_2)

1.7 110

$[0,2]$ (of type BC) $x_1 \in k^*$, $x_2 = 1$, :

$$t_2(c_1) = t_1(\frac{-c_2 x_1}{a_3})t_2(-a_3 b_3 x_1) \Leftrightarrow x_1 = -\frac{c_1}{a_3 b_3} = -\frac{a_3}{c_2},\ c_1 c_2 = a_3^2 b_2$$

$$\sum \psi(u_1(\frac{-1}{(b_3 x_1)})u_2(-x_1)) = \phi(\frac{a_3}{c_1} + \frac{a_3}{c_2})$$

Should equal (1.7) (symmetric in c_1 and c_2)

1.8 123

$[1,2]$ (of type AA) $x_1 \in k$, $x_2 \in k$, :

$$t_2(c_1) = t_2(-d_2) \Leftrightarrow c_1 = -d_2$$

$$\sum \psi(1) = q^2$$

Should equal (1.14)

1.9 120

$[1,0]$ (of type CB) $x_1 = 1$, $x_2 \in k^*$, :

$$t_2(c_1) = t_1(\frac{-1}{(a_3 x_2)})t_2(a_3 b_3 d_2 x_2^2) \Leftrightarrow x_2 = -\frac{1}{a_3},\ c_1 = \frac{b_3}{a_3} d_2$$

$$\sum \psi(u_2(\frac{-1}{(d_2 x_2)})) = \phi(\frac{a_3}{d_2}) = \phi(\frac{b_3}{c_1})$$

Should equal (1.15)

1.10 101

$[0,2]$ (of type BA) $x_1 \in k^*$, $x_2 \in k$, :

$$t_2(c_1) = t_1(b_2x_1)t_2(-a_2c_3x_1) \Leftrightarrow x_1 = \frac{1}{b_2},\, c_1 = -\frac{a_2}{b_2}c_3$$

$$\sum \psi(u_1(x_2 - 1/(c_3x_1) - x_2/(b_2x_1))) = \sum \phi(-\frac{b_2}{c_3}) = q\phi(-\frac{b_2}{c_3}) = q\phi(\frac{a_2}{c_1})$$

Should equal (1.22)

1.11 102

$[1,0]$ (of type AB) $x_1 \in k$, $x_2 \in k^*$, :

$$t_2(c_1) = t_1(-b_2/(d_3x_2))t_2(-a_2d_3x_2^2) \Leftrightarrow x_2 = -\frac{b_2}{d_3},\, c_1d_3 = -a_2b_2^2$$

$$\sum \psi(u_1(-x_1 + x_2)u_2(x_1 + 1/(a_2x_2))) = \sum_{x_1} \phi(x_2 + \frac{1}{a_2x_2}) = \sum_{x_1} \phi(-\frac{b_2}{d_3} - \frac{d_3}{a_2b_2}) = q\phi(\frac{b_2}{c_1} - \frac{b_2}{d_3})$$

Should equal (1.23) - true since $\frac{d_3}{a_2b_2} = -\frac{b_2}{c_1}$

1.12 100

$[0,0]$ (of type BB) $x_1 \in k^*$, $x_2 \in k^*$, :

$$t_2(c_1) = t_1(b_2x_1/(a_3x_2))t_2(-a_2a_3b_3x_1x_2^2) \Rightarrow x_1 = \frac{a_3}{b_2}x_2,\, x_2^3 = -\frac{b_2c_1}{a_2a_3^2b_3},$$

with sum

$$\sum \psi(u_1(-x_2 - 1/(b_3x_1) - 1/(b_2x_1))u_2(-x_1 - 1/(a_2x_2)))$$
$$= \sum \phi(-x_2 - \frac{1}{b_3x_1} - \frac{1}{b_2x_1} - x_1 - \frac{1}{a_2x_2})$$
$$= \mathcal{S}_3\left(\frac{b_2c_1}{a_2a_3^2b_3}; 1 + \frac{a_3}{b_2}, \frac{1}{a_2} + \frac{1}{a_3} + \frac{b_2}{a_3b_3}\right).$$

Should equal (1.24) - true, just permute the indices 1 and 2 in a_i, b_j, c_k.

1.13 232

$[2,1]$ (of type CC) $x_1 = 1$, $x_2 = 1$, :

$$t_1(d_1) = t_1(d_3) \Leftrightarrow d_1 = d_3$$

$$\sum \psi(1) = 1$$

Should equal (1.3)

1.14 213

$[2,1]$ (of type AA) $x_1 \in k$, $x_2 \in k$, :

$$t_1(d_1) = t_1(-c_2) \Leftrightarrow c_2 = -d_1$$

$$\sum \psi(1) = q^2$$

Should equal (1.8)

1.15 210

$[2,0]$ (of type CB) $x_1 = 1$, $x_2 \in k^*$, :

$$t_1(d_1) = t_1(a_3 b_3 c_2 x_2^2) t_2(\frac{-1}{(b_3 x_2)}) \Leftrightarrow x_2 = -\frac{1}{b_3},\ d_1 = \frac{a_3}{b_3} c_2$$

$$\sum \psi(u_1(\frac{-1}{(c_2 x_2)})) = \phi(\frac{b_3}{c_2}) = \phi(\frac{a_3}{d_1})$$

Should equal (1.9)

1.16 221

$[2,1]$ (of type AC) $x_1 \in k$, $x_2 = 1$, :

$$t_1(d_1) = t_1(-c_3) t_2(-d_2/c_3) \Leftrightarrow c_3 = -d_1,\ c_3 = -d_2$$

$$\sum \psi(u_1(x_1) u_2(-x_1)) = \sum_{x_1} \phi(1) = q$$

Should equal (1.16)

1.17 220

$[0,1]$ (of type BC) $x_1 \in k^*$, $x_2 = 1$, :

$$t_1(d_1) = t_1(-a_3 b_3 x_1) t_2(-d_2 x_1/b_3) \Leftrightarrow x_1 = -\frac{d_1}{a_3 b_3} = \frac{-b_3}{d_2},\ d_1 d_2 = a_3 b_3^2$$

$$\sum \psi(u_1(-x_1) u_2(-1/(a_3 x_1))) = \phi(\frac{b_3}{d_2} + \frac{b_3}{d_1})$$

Should equal (1.17)

1.18 201

$[2,0]$ (of type AB) $x_1 \in k$, $x_2 \in k^*$, :

$$t_1(d_1) = t_1(-b_2 c_3 x_2^2) t_2(-a_2/(c_3 x_2)) \Leftrightarrow x_2 = -\frac{a_2}{c_3},\ c_3 d_1 = -a_2^2 b_2$$

$$\sum \psi(u_1(x_1 + 1/(b_2 x_2)) u_2(-x_1 + x_2)) = \sum_{x_1} \phi(-\frac{c_3}{a_2 b_2} - \frac{a_2}{c_3}) = q\phi(\frac{a_2}{d_1} - \frac{a_2}{c_3})$$

Should equal (1.25)

1.19 202

$[0,1]$ (of type BA) $x_1 \in k^*$, $x_2 \in k$, :

$$t_1(d_1) = t_1(-b_2 d_3 x_1) t_2(a_2 x_1) \Leftrightarrow x_1 = -\frac{d_1}{b_2 d_3} = \frac{1}{a_2},\ b_2 d_3 = -d_1 a_2$$

$$\sum \psi(u_2(x_2 - 1/(d_3 x_1) - x_2/(a_2 x_1))) = \sum_{x_2} \phi(x_2 + \frac{b_2}{d_1} - x_2) = q\phi(\frac{b_2}{d_1})$$

Should equal (1.26)

1.20 200

$[0,0]$ (of type BB) $x_1 \in k^*$, $x_2 \in k^*$, :

$$t_1(d_1) = t_1(-a_3 b_2 b_3 x_1 x_2^2) t_2(a_2 x_1/(b_3 x_2)) \Leftrightarrow x_1 = \frac{b_3}{a_2} x_2,\ x_2^3 = -\frac{a_2 d_1}{a_3 b_2 b_3^2}$$

and the sum is

$$\begin{aligned}\sum \psi(u_1(-x_1 - 1/(b_2 x_2)) u_2(-x_2 - 1/(a_3 x_1) - 1/(a_2 x_1))) = \\ = \sum \phi\left(-x_1 - \frac{1}{b_2 x_2} - x_2 - \frac{1}{a_3 x_1} - \frac{1}{a_2 x_1}\right) \\ = \mathcal{S}_3\left(\frac{a_2 d_1}{a_3 b_2 b_3^2}; 1 + \frac{b_3}{a_2}, \frac{1}{b_2} + \frac{1}{b_3} + \frac{a_2}{a_3 b_3}\right).\end{aligned}$$

Should equal (1.27) - true, just permute the indices 1 and 2 in a_i, b_j, d_h.

1.21 030

$[1,2,1]$ (of type CCC) $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, :

$$t_1(a_1) t_2(b_1) = t_1(a_3) t_2(b_3) \Leftrightarrow b_1 = b_3,\ a_1 = a_3$$

$$\sum \psi(1) = 1$$

Should equal (1.4)

1.22 011

$[1,0,1]$ (of type CBA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, :

$$t_1(a_1) t_2(b_1) = t_1(-c_2 x_2) t_2(c_3 x_2) \Leftrightarrow x_2 = -\frac{a_1}{c_2} = \frac{b_1}{c_3}$$

$$\sum \psi(u_2(-x_2)) = \sum \phi(\frac{a_1}{c_2}) = q\phi(\frac{a_1}{c_2}) = q\phi(-\frac{b_1}{c_3})$$

Should equal (1.10)

1.23 012

$[1,2,0]$ (of type ACB) $x_1 \in k$, $x_2 = 1$, $x_3 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(-c_2d_3x_3^2)t_2(1/x_3) \Leftrightarrow x_3 = \frac{1}{b_1},\ c_2d_3 = -a_1b_1^2$$

$$\sum \psi(u_1(-x_1 + 1/(c_2x_3))u_2(x_1 - 1/(d_3x_3))) = \sum_{x_1} \phi((\frac{1}{c_2} - \frac{1}{d_3})\frac{1}{x_3}) = \sum_{x_1} \phi(\frac{b_1}{c_2} - \frac{b_1}{d_3}) = q\phi(\frac{b_1}{c_2} - \frac{b_1}{d_3})$$

Should equal (1.11)

1.24 010

$[0,2,0]$ (of type BCB) $x_1 \in k^*$, $x_2 = 1$, $x_3 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(-a_3c_2x_3^2/x_1)t_2(-b_3x_1^2/x_3) \Leftrightarrow x_1 = -\frac{a_3c_2x_3^2}{a_1},\ x_3^3 = -\frac{a_1^2b_1}{a_3^2b_3c_2^2} = -\frac{a_1^3}{c_2^3}\frac{b_1c_2}{a_1a_3^2b_3},$$

with sum

$$\begin{aligned}&\sum \psi(u_1(-1/(c_2x_3) - 1/(b_3x_1))u_2(-x_1 + x_3/x_1 + x_1/(a_3x_3))) \\ &= \sum \phi(-\frac{1}{c_2x_3} - \frac{1}{b_3x_1} - x_1 + \frac{x_3}{x_1} + \frac{x_1}{a_3x_3}) \\ &= \mathcal{S}_3\left(\frac{b_1c_2}{a_1a_3^2b_3}; 1 + \frac{a_3}{b_1}, \frac{1}{a_1} + \frac{1}{a_3} + \frac{b_1}{a_3b_3}\right).\end{aligned}$$

Should equal (1.12) - true, just permute the indices 1 and 2 in a_i, b_j, c_k.

1.25 021

$[0,2,1]$ (of type BAC) $x_1 \in k^*$, $x_2 \in k$, $x_3 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(1/x_1)t_2(-c_3d_2x_1^2) \Leftrightarrow x_1 = \frac{1}{a_1},\ c_3d_2 = -a_1^2b_1$$

$$\sum \psi(u_1(x_2 - 1/(c_3x_1))u_2(-x_2 + 1/(d_2x_1))) = \sum_{x_2} \phi(-\frac{a_1}{c_2} + \frac{a_1}{d_2}) = q\phi(\frac{a_1}{d_2} - \frac{a_1}{c_2})$$

Should equal (1.18)

1.26 022

$[1,0,1]$ (of type ABC) $x_1 \in k$, $x_2 \in k^*$, $x_3 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(-d_3x_2)t_2(d_2x_2) \Leftrightarrow x_2 = -\frac{a_1}{d_3} = \frac{b_1}{d_2},\ b_1d_3 = -a_1d_2$$

$$\sum \psi(u_1(x_2)) = \sum_{x_1} \phi(\frac{b_1}{d_2}) = q\phi(\frac{b_1}{d_2})$$

Should equal (1.19)

1.27 020

$[0,0,1]$ (of type BBC) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 = 1$, :

$$t_1(a_1)t_2(b_1) = t_1(a_3x_2/x_1)t_2(-b_3d_2x_1^2x_2) \Leftrightarrow x_2 = \frac{a_1}{a_3}x_1,\ x_1^3 = -\frac{a_3b_1}{a_1b_3d_2}$$

with sum

$$\sum \psi(u_1(-x_1 - x_2 - 1/(b_3x_1))u_2((a_3 - d_2x_1)/(a_3d_2x_1x_2))) =$$
$$= \sum \phi\left(-x_1 - x_2 - \frac{1}{b_3x_1} + \frac{1}{d_2x_1x_2} - \frac{1}{a_3x_2}\right)$$
$$= \mathcal{S}_3\left(\frac{a_3b_1}{a_1b_3d_2}; 1 + \frac{b_3}{a_1}; \frac{1}{b_1} + \frac{1}{b_3} + \frac{a_1}{a_3b_3}\right).$$

Should equal (1.20) - true, just permute the indices 1 and 2 in a_i, b_j, d_h.

1.28 003

$[1,2,1]$ (of type AAA) $x_1 \in k$, $x_2 \in k$, $x_3 \in k$, :

$$t_1(a_1)t_2(b_1) = t_1(b_2)t_2(a_2) \Leftrightarrow a_1 = b_2,\ b_1 = a_2$$
$$\sum \psi(1) = q^3$$

Should equal (1.28)

1.29 001

$[0,2,0]$ (of type BAB) $x_1 \in k^*$, $x_2 \in k$, $x_3 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(b_2x_3^2/x_1)t_2(-a_2c_3x_1^2/x_3) \Leftrightarrow x_1 = \frac{b_2}{a_1}x_3^2,\ x_3 = -\frac{a_2}{b_1}c_3x_1^2$$

so $x_1^3 = \frac{a_1b_1^2}{a_2^2b_2}\frac{1}{c_3^2}$ and $x_3^3 = -\frac{a_1^2b_1}{a_2b_2^2}\frac{1}{c_3}$, with sum

$$\sum_{x_1,x_2,x_3 \text{ as above}} \psi(u_1(x_2 - 1/(c_3x_1) + 1/(b_2x_3))u_2(x_1/x_3 - x_2 + x_3 + x_3/(a_2x_1)))$$
$$= q \sum_{x_1,x_3 \text{ as above}} \phi(-\frac{1}{c_3x_1} + \frac{1}{b_2x_3} + \frac{x_1}{x_3} + x_3 + \frac{x_3}{a_2x_1})$$
$$= \mathcal{S}_3\left(\frac{a_1^2b_1}{a_2b_2^2}\frac{1}{c_3}; -1 - \frac{b_2}{a_1} - \frac{a_2b_2}{a_1b_1}, -\frac{1}{b_2} - \frac{a_1}{a_2b_2}\right).$$

Should equal (1.29) – we notice that

$$(a_1, a_2, b_1, b_2, c_3) \mapsto (b_1, b_2, a_1, a_2, d_3)$$

makes the result of (1.29) into that of (1.30).

1.30 002

$[1, 0, 0]$ (of type ABB) $x_1 \in k$, $x_2 \in k^*$, $x_3 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(-b_2d_3x_2x_3^2)t_2(-a_2x_2/x_3)$$

We have then $a_1 = -b_2d_3x_2x_3^2$ and $b_1 = -a_2x_2/x_3$, which implies $x_2 = -\frac{b_1}{a_2}x_3$ and $x_3^3 = \frac{a_1a_2}{b_1b_2d_3} = \frac{a_2^3}{b_1^3}\frac{a_1b_1^2}{a_2^2b_2d_3}$. The sum is

$$\sum_{x_1,x_2,x_3 \text{ as above}} \psi(u_1(-x_1 + x_2 - 1/(b_2x_3))u_2(x_1 - x_3 + 1/(d_3x_2x_3) + 1/(a_2x_2)))$$

$$= q \sum_{x_2,x_3 \text{ as above}} \phi(x_2 - \frac{1}{b_2x_3} - x_3 + \frac{1}{d_3x_2x_3} + \frac{1}{a_2x_2})$$

$$= \mathcal{S}_3\left(\frac{a_1b_1^2}{a_2^2b_2d_3}; -1 - \frac{a_2}{b_1} - \frac{a_2b_2}{a_1b_1}; -\frac{1}{a_2} - \frac{b_1}{a_2b_2}\right).$$

Should equal (1.29) – we notice that

$$(b_1, b_2, a_1, a_2, d_3) \mapsto (a_1, a_2, b_1, b_2, c_3)$$

makes the result of (1.30) into that of (1.29).

1.31 000

$[0, 0, 0]$ (of type BBB) $x_1 \in k^*$, $x_2 \in k^*$, $x_3 \in k^*$, :

$$t_1(a_1)t_2(b_1) = t_1(a_3b_2x_2x_3^2/x_1)t_2(a_2b_3x_1^2x_2/x_3)$$

We have then $a_1 = a_3b_2x_2x_3^2/x_1$ and $b_1 = a_2b_3x_1^2x_2/x_3$, hence $x_3 = \frac{a_2b_3}{b_1}x_1^2x_2$ and $(x_1x_2)^3 = \frac{a_1b_1^2}{a_2^2a_3b_2b_3^2}$. The sum is

$$\sum \psi(u_1(-x_2 - 1/(b_3x_1) - 1/(b_2x_3))u_2(-(x_1 + x_3)(a_2a_3x_1x_2x_3 + a_2x_1 + a_3x_3)/(a_2a_3x_1x_2x_3)))$$

$$= \sum_{t \in k^*}\sum_{\zeta \in \mu_3} \phi(-x_2 - \frac{1}{b_3x_1} - \frac{1}{b_2x_3} - x_1 - x_3 - \frac{x_1}{a_3x_2x_3} - \frac{1}{a_2x_2} - \frac{1}{a_3x_2} - \frac{x_3}{a_2x_1x_2})$$

$$= \sum_{\zeta|\beta^3 = \frac{a_1b_1^2}{a_2^2a_3b_2b_3^2}} \mathcal{K}\left(-1 - \frac{a_2b_2b_3}{a_1b_1}\zeta - \frac{1}{a_2\zeta} - \frac{1}{a_3\zeta} - \frac{b_3}{b_1} - \frac{a_2b_3}{b_1}\zeta; -\beta - \frac{1}{b_3} - \frac{b_1}{a_2b_2b_3\zeta}\right).$$

$[1, 0, 1]$ (of type CBA) $x_1 = 1$, $x_2 \in k^*$, $x_3 \in k$, :

$$t_1(a_1)t_2(b_1) = t_1(-a_3b_2x_2)t_2(a_2b_3x_2)$$

We have that $a_1 = -a_3 b_2 x_2$ and $b_1 = a_2 b_3 x_2$. Hence $x_3 = -a_1/(a_3 b_2) = b_1/(a_2 b_3)$, and we see that if $b_1 \neq -a_1 a_2 b_3/(a_3 b_2)$ then the value of the sum is zero. Suppose now that $b_1 = -a_1 a_2 b_3/(a_3 b_2)$. We rewrite the sum as

$$\sum \psi(u_2(x_3 + x_3/(a_3 x_2) - x_3/(a_2 x_2))) \\ = \sum_{x_3 \in k} \phi\left(x_3\left(\frac{b_2 b_3 - a_1 b_3 - b_1 b_2}{b_2 b_3}\right)\right).$$

If $a_1 = \frac{b_2 b_3 - b_1 b_2}{b_3}$, then the value of the sum is q. Otherwise, it is equal to 0.

Should equal (1.31)