On characters of a Sylow p-subgroup of $G(p^f)$

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Notation

- $q = p^f$ with p a prime
- $G = G(q) = \mathrm{Y}_r(q)$ split finite reductive group of type Y and rank r defined over $\mathbb{F}_q$ (eg. $\mathrm{A}_r = \mathrm{SL}_{r+1}(q)$), with Φ^+ poset of positive roots
- $U = U(q) = \mathrm{UY}_r(q) \in \mathrm{Syl}_p(G)$
- $\mathrm{Irr}(U(q))$ the set of ordinary irreducible characters of $U(q)$, and $k(U(q)) =$ number of conjugacy classes in $U(q) = |\,\mathrm{Irr}(U(q))|$

Goal: get a parametrization of $\mathrm{Irr}(U(q))$ *in a uniform way* in both p and f – a *generic* parametrization.

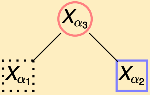
Outline

1) Partition of $\mathrm{Irr}(U(q))$ into sets $\mathrm{Irr}(U(q))_\Sigma$ simpler to study, for $\Sigma \subseteq \Phi^+$
2) Parametrization of $\mathrm{Irr}(U(q))_\Sigma$ via CHEVIE/GAP4. Straightforward, up to important special configurations called *nonabelian cores*. At most 10 of them if $r \leq 4$, hundreds of them if $\mathrm{Y} = \mathrm{E}_6$
3) More into-depth study of nonabelian cores. Hopefully more machine help

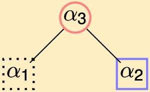
- Reduction algorithm to parametrize $\mathrm{Irr}(U(q))$, based on the following.

Example: $\mathrm{Irr}(\mathrm{UA}_2(q))$. ○ **= nontrivial central,** □ **= inflation,** ⬚ **= induction**

Let $V = \mathrm{UA}_2(q) = X_{\alpha_1}X_{\alpha_2}X_{\alpha_3}$, with $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$.



$[x_{\alpha_1}(s), x_{\alpha_2}(t)] = x_{\alpha_3}(st)$



$\alpha_1 + \alpha_2 = \alpha_3$

- Let $1_{X_{\alpha_3}} \neq \lambda \in \mathrm{Irr}(X_{\alpha_3})$. The inertia subgroup
$$I_V(\mathrm{Infl}_{X_{\alpha_3}}^{X_{\alpha_2}X_{\alpha_3}}(\lambda)) = X_{\alpha_2}X_{\alpha_3}.$$
Then $\mathrm{Ind}_{X_{\alpha_2}X_{\alpha_3}}^{V}\left(\mathrm{Infl}_{X_{\alpha_3}}^{X_{\alpha_2}X_{\alpha_3}}(\lambda)\right) \in \mathrm{Irr}(V)$. These give $q-1$ *irreducible characters of degree* q for distinct choices of $\lambda \in \mathrm{Irr}(X_{\alpha_3}) \setminus \{1_{X_{\alpha_3}}\}$.
- $[V, V] = X_{\alpha_3}$. Get q^2 *linear characters* by inflating from $V/X_{\alpha_3} \cong \mathbb{F}_q^2$.

From now on, we assume p is *not a very bad prime* for U, so that
$$\alpha + \beta = \gamma \in \Phi^+ \iff [X_\alpha, X_\beta] \neq 1.$$

Patterns and quatterns

- $\mathcal{P} \subseteq \Phi^+$ *pattern* if $\alpha, \beta \in \mathcal{P}$, $\alpha + \beta \in \Phi^+ \Rightarrow \alpha + \beta \in \mathcal{P}$.

$$\mathcal{P} \subseteq \Phi^+ \text{ pattern } \Leftrightarrow X_{\mathcal{P}} := \prod_{\alpha \in \mathcal{P}} X_\alpha \text{ subgroup of } U.$$

- $\mathcal{K}$ *normal* in the pattern $\mathcal{P}$ (or $\mathcal{K} \trianglelefteq \mathcal{P}$) if $\alpha \in \mathcal{K}$, $\beta \in \mathcal{P}$, $\alpha + \beta \in \mathcal{P} \Rightarrow \alpha + \beta \in \mathcal{K}$.

$$\mathcal{K} \subseteq \mathcal{P} \text{ patterns. Then } \mathcal{K} \trianglelefteq \mathcal{P} \Leftrightarrow X_{\mathcal{K}} \trianglelefteq X_{\mathcal{P}}.$$

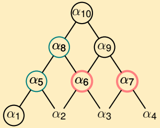
- $\mathcal{S} \subseteq \Phi^+$ *quattern* if $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, with $\mathcal{P}$ pattern, $\mathcal{K} \trianglelefteq \mathcal{P}$.

$$X_{\mathcal{S}} := X_{\mathcal{P}} / X_{\mathcal{K}} \text{ \textit{quattern group} wrt } \mathcal{S}.$$

- *Central root support* $\mathcal{Z}(\mathcal{S})$ of $\mathcal{S}$, s.t. $Z(X_{\mathcal{S}}) = X_{\mathcal{Z}(\mathcal{S})}$,

$$\mathcal{Z}(\mathcal{S}) = \{\gamma \in \mathcal{S} \mid \gamma + \alpha \notin \mathcal{S} \text{ for all } \alpha \in \mathcal{S}\}.$$

E.g.: $U = \mathrm{UA}_4(q)$



$\mathcal{Q} := \{\alpha_1, \alpha_9, \alpha_{10}\}$ pattern (not normal in $\mathcal{P} := \Phi^+$)

$\mathcal{K} := \mathcal{Q} \cup \{\alpha_5, \alpha_8\} \trianglelefteq \mathcal{P}$, $\mathcal{P} = \Phi^+$

$\mathcal{S} := \Phi^+ \setminus \mathcal{K}$ quattern, $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$

$\mathcal{Z}(\mathcal{S}) = \{\alpha_6, \alpha_7\}$

Standard quatterns

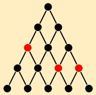
- Denote by Σ an *antichain* in Φ^+,

$$\gamma_1, \gamma_2 \in \Sigma \Longrightarrow \gamma_1 \npreceq \gamma_2 \text{ and } \gamma_2 \npreceq \gamma_1.$$

- The *standard quattern* wrt Σ is $\mathcal{S}_\Sigma := \Phi^+ \setminus \mathcal{K}_\Sigma$, where

$$\mathcal{K}_\Sigma := \{\delta \in \Phi^+ \mid \delta \npreceq \gamma \text{ for every } \gamma \in \Sigma\}.$$

E.g.: $U = \mathrm{UA}_5(q)$



$\Sigma = \mathcal{Z}(\mathcal{S}_\Sigma)$

- $\mathcal{S} \subseteq \Phi^+$ quattern, $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$. The *characters of $X_{\mathcal{S}}$ supported on $\mathcal{Z}$* are
$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} := \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\gamma \not\subseteq \ker(\chi) \text{ for every } \gamma \in \mathcal{Z}\}.$$
- For an antichain $\Sigma \subseteq \Phi^+$, define the *characters of U supported on Σ*,
$$\mathrm{Irr}(U)_\Sigma := \mathrm{Infl}^{U}_{X_{\mathcal{S}_\Sigma}}\left(\mathrm{Irr}(X_{\mathcal{S}_\Sigma})_\Sigma\right).$$

Proposition (Himstedt, Le, Magaard ’15)

$$\mathrm{Irr}(U) = \bigsqcup_{\Sigma \text{ antichain in } \Phi^+} \mathrm{Irr}(U)_\Sigma.$$

- How to study $\mathrm{Irr}(U)_\Sigma$?

Proposition ($\mathrm{Infl}_\beta := \mathrm{Infl}_K^{KX_\beta}$, $\mathrm{Ind}_\alpha := \mathrm{Ind}_K^{KX_\alpha}$)

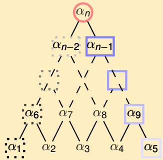
Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ quattern, $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$. Assume $\alpha, \beta \in \mathcal{S}$ with

- $\alpha + \beta = \gamma \in \mathcal{Z}$;
- β is *well-behaved*, that is $\delta + \beta \notin \mathcal{S}$ for all $\delta \in \mathcal{S} \setminus \{\alpha\}$.

If $\mathcal{S}' := \mathcal{P}' \setminus \mathcal{K}'$, with $\mathcal{P}' = \mathcal{P} \setminus \{\alpha\}$ and $\mathcal{K}' = \mathcal{K} \cup \{\beta\}$, then

$$\mathrm{Ind}^\alpha \, \mathrm{Infl}_\beta : \mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}} \xrightarrow{\sim} \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}.$$

E.g.: $\mathrm{Irr}(\mathrm{UA}_r(q))_{\{\alpha_n\}}$



- Call a quattern $\mathcal{S}$ a *core* if the previous reduction cannot be applied.

If $\mathcal{S}$ is an abelian core (i.e. $X_\mathcal{S}$ abelian), then $\mathrm{Irr}(U)_\Sigma$ is fully parametrized!

- Numbers of nonabelian cores in each type. **How to study them?**

Y_r, $r \leq 3$
0

B_4	C_4	D_4	F_4
1	0	1	6

B_5	C_5	D_5
7	1	7

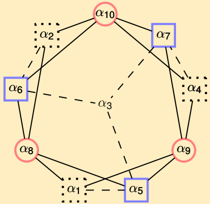
B_6	C_6	D_6	E_6
36	16	31	109

E.g. (Himstedt, Le, Magaard '11): $\mathrm{UD}_4(q)$, $\Sigma = \{\alpha_8, \alpha_9, \alpha_{10}\}$, $\mathcal{S}_\Sigma = \{\alpha_1, \dots, \alpha_{10}\}$

Join α and β with γ if $\alpha + \beta = \gamma \in \mathcal{S}$. Let $\lambda \in \mathrm{Irr}(Z)_\Sigma$.

$$I_{X_{\{\alpha_1,\alpha_2,\alpha_4\}}}(\mathrm{Infl}_{\{\alpha_5,\alpha_6,\alpha_7\}} \lambda) \cong \begin{cases} 1, & \text{if } p \geq 3, \\ \mathbb{F}_q, & \text{if } p = 2 \end{cases}.$$

If $p = 2$, different expression of $k(U(q)) \in \mathbb{Z}_{\geq 0}[q-1]$.



Theorem (Goodwin, Le, Magaard, P. '16)

If $r \leq 4$ and p is not very bad for Φ^+, then each $\chi \in \mathrm{Irr}(\mathrm{UY}_r(q))$ is constructed as an inflation, followed by an induction of a character of a quattern group of $\mathrm{UY}_r(q)$. This yields a parametrization of $\mathrm{Irr}(\mathrm{UY}_r(q))$ in these cases.

$\mathcal{S}$ quattern. Let $\mathcal{R}(\mathcal{S})$ the set of relations $\alpha + \beta = \gamma$ with $\alpha, \beta, \gamma \in \mathcal{S}$. Assume

$$|\mathcal{S}| = |\mathcal{S}'|, \quad |\mathcal{Z}(\mathcal{S})| = |\mathcal{Z}(\mathcal{S}')|, \quad |\mathcal{R}(\mathcal{S})| = |\mathcal{R}(\mathcal{S}')|, \qquad \text{(P)}$$

for $\mathcal{S}$, $\mathcal{S}'$ nonabelian cores of U. Does this imply $X_{\mathcal{S}} \cong X_{\mathcal{S}'}$?

E.g.: D_7, $|\mathcal{S}| = |\mathcal{S}'| = 10$, $|\mathcal{Z}(\mathcal{S})| = |\mathcal{Z}(\mathcal{S}')| = 3$, $|\mathcal{R}(\mathcal{S})| = |\mathcal{R}(\mathcal{S}')| = 9$

$X_{\mathcal{S}}$ isomorphic to quattern group $X_{\mathcal{S}_\Sigma}$ in D_4, $\Sigma = \{\alpha_8, \alpha_9, \alpha_{10}\}$, but $X_{\mathcal{S}} \ncong X_{\mathcal{S}'}$.

Proposition A

Let $\mathcal{S}$, $\mathcal{S}'$ two quatterns in type Υ_r, $r \leq 6$, which satisfy (P). Then $X_{\mathcal{S}} \cong X_{\mathcal{S}'}$.

Reduce numbers of nonabelian cores to those of their isomorphism classes.

	D_5	D_6	E_6	D_7	E_7
Nonab. cores	7	31	109	169	3392
Iso. classes	2	7	16	≤ 35	≤ 393

Theorem B

If $U = \mathrm{UD}_6(q)$ or $U = \mathrm{UE}_6(q)$, then $\mathrm{Irr}(U)$ is parametrized for every prime p.

In particular, the numbers $k(U, q^d)$ of characters of fixed degree q^d coincide with the ones obtained in [Goodwin, Mosch, Röhrle '15] if p is large enough.