Quattern groups and characters of finite unipotent groups

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Lausanne, October 18th 2016

Some notation

- $q = p^d$ with p a prime
- $G = G(q) = \mathrm{Y}_r(q)$ a split finite reductive group of type Y and rank r defined over $\mathbb{F}_q$ (eg. $\mathrm{A}_r(q) = \mathrm{SL}_{r+1}(q), \mathrm{G}_2(q), \dots$)
- $U = U(q) = \mathrm{UY}_r(q) \in \mathrm{Syl}_p(G)$
- $\mathrm{Irr}(U(q))$ the set of ordinary irreducible characters of $U(q)$, and $k(U(q)) := |\mathrm{Irr}(U(q))|$.

Research problem: get a parametrization of the set $\mathrm{Irr}(U(q))$ *in a uniform way* in both p and d – also called *generic* parametrization.

Content of the talk

The above is achieved when:

- G is of rank 4 (or less), and
- p is not a very bad prime for G. ($p \neq 2$ in types $\mathrm{B}_r, \mathrm{C}_r$ and F_4, $p \neq 2, 3$ in type G_2),

via a reduction algorithm to successive subquotients of $U(q)$ called *quattern groups*.

Motivation

- **Modular decomposition numbers**. Let $\ell \neq p$ be a prime, $\ell \mid |G|$. Replace generalized Gelfand-Graev representations (GGGRs) in bad characteristics by $\mathrm{Ind}_U^G(\chi)$, where $\chi \in \mathrm{Irr}(U)$, to get ℓ-modular decomposition numbers.
- **Generic character tables**. Get labels for generic irreducible characters and conjugacy classes, compute character values as a function of q and the labels. Recent progress: [Goodwin, Le, Magaard '16] for $\mathrm{UD}_4(q)$, [Sun '16] for $\mathrm{U}\,{}^3\mathrm{D}_4(q^3)$.
- **Character degrees**. We want to determine the set

$$\mathrm{cd}(U(q)) = \{\chi(1) \mid \chi \in \mathrm{Irr}(U(q))\}.$$

  In particular, is it true that

$$\chi(1) = q^d \text{ for every } \mathrm{Irr}(U(q)) \iff p \text{ is a good prime for } G?$$

- **Conjectures by Isaacs, Lehrer, Higman**. Statement:

  p good prime for $G \Rightarrow$ the number $k(U(q), q^d)$ of irreducible characters of $U(q)$ of degree q^d is a polynomial in $q-1$ with coefficients in $\mathbb{Z}_{\geq 0}$.

  Not even known for $k(\mathrm{UA}_r(q))$! Look at exceptional groups.

Some important results

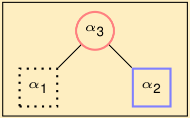
- [Isaacs '95]: every irreducible character of $\mathrm{UA}_r(q)$ is a power of q. Proved by inductive properties inside *algebra groups*, that is, groups of the form $1 + J$, with J a nilpotent algebra over $\mathbb{F}_q$.
- Refinements of the above:
  - [Sangroniz '03]: every character of $\mathrm{UY}_r(q)$, for Y classical, is a power of q if and only if $p \neq 2$.
  - [Evseev '11]: full parametrization of $\mathrm{Irr}(\mathrm{UA}_r(q))$ for every $r \leq 12$.
- [Marberg '11]: relying on the general theory of supercharacters in [Diaconis, Isaacs'08] and [Evseev '11], get
$$k(\mathrm{UA}_r(q), q^d) \in \mathbb{Z}_{\geq 0}[q-1] \text{ for every } d \leq 8.$$
- [Himstedt, Le, Magaard '11]: using representable sets and Clifford and Gallagher theory, $\mathrm{Irr}(\mathrm{UD}_4(q))$ is parametrized. The parametrization is uniform for every odd q, but this uniformity stops when $p = 2$ is a bad prime.

Our approach algorithmically develops [Himstedt, Le, Magaard '11]. Implementation using CHEVIE.

- Algorithm: iteratively based on $V := \mathrm{UA}_2(q)$. Define the root elements

$$x_{\alpha_1}(t_1) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad x_{\alpha_2}(t_2) = \begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}, \quad x_{\alpha_3}(t_3) = \begin{pmatrix} 1 & & t_3 \\ & 1 & \\ & & 1 \end{pmatrix},$$

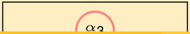
  and the *root subgroups* $X_{\alpha_i} := \{x_{\alpha_i}(t) \mid t \in \mathbb{F}_q\}$ for $i = 1, 2, 3$. We have $V = X_{\alpha_1} X_{\alpha_2} X_{\alpha_3}$, $[x_{\alpha_1}(s), x_{\alpha_2}(t)] = x_{\alpha_3}(st)$ and $Z(V) = [V, V] = X_{\alpha_3}$.

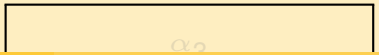


- Let $\chi \in \mathrm{Irr}(V)$ with $X_{\alpha_3} \not\subseteq \ker(\chi)$. Let $\lambda := \chi|_{X_{\alpha_3}}$. We put $\psi := \mathrm{Inf}_{X_{\alpha_3}}^{X_{\alpha_2} X_{\alpha_3}}(\lambda) \in \mathrm{Irr}(X_{\alpha_2} X_{\alpha_3})$. The inertia subgroup is

$$I_V(\psi) = \{v \in V \mid \psi^v = \psi\} = X_{\alpha_2} X_{\alpha_3}.$$

  We get $\mathrm{Ind}_{X_{\alpha_2} X_{\alpha_3}}^{V}(\psi) \in \mathrm{Irr}(V)$. This yields $q - 1$ characters of degree q.

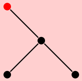




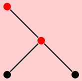
Very bad primes: a different situation

The situation is very different for $\mathrm{UB}_2(q) = X_{\alpha_1} \cdots X_{\alpha_4}$. Here

$$[x_{\alpha_1}(s), x_{\alpha_2}(t)] = x_{\alpha_3}(st)x_{\alpha_4}(st^2) \quad \text{and} \quad [x_{\alpha_2}(t), x_{\alpha_3}(u)] = x_{\alpha_4}(\mathbf{2}tu).$$



$p \neq 2$. Then $Z(\mathrm{UB}_2(q)) = X_{\alpha_4}$, $k(\mathrm{UB}_2(q)) = 2q^2 - 1$.



$p = 2$. Then $Z(\mathrm{UB}_2(q)) = X_{\alpha_3}X_{\alpha_4}$, $k(\mathrm{UB}_2(q)) = 5q^2 - 6q + 2$.

From now on, we assume p is *not a very bad prime* for G, i.e. p does *not* divide the coefficients involved in the Chevalley relations (in bold above).

Further notation

- We denote by Φ^+ the poset of positive roots, with order defined by $\alpha < \beta$ for $\alpha, \beta \in \Phi$ if and only if $\beta - \alpha$ is a sum of positive roots
- $x_\alpha(t)$ (resp. X_α) is a root element (resp. root subgroup)
- Let K be a subquotient of U. We denote $\mathrm{Ind}^\alpha = \mathrm{Ind}_K^{KX_\alpha}$ and $\mathrm{Inf}_\beta = \mathrm{Inf}_K^{KX_\beta}$.

- Let $\mathcal{P} \subseteq \Phi^+$. Say $\mathcal{P}$ *pattern* (or *closed*) if
$$\alpha, \beta \in \mathcal{P},\ \alpha + \beta \in \Phi^+ \Longrightarrow \alpha + \beta \in \mathcal{P}.$$
- Fact 1: $\mathcal{P} \subseteq \Phi^+$ pattern $\Longleftrightarrow X_{\mathcal{P}} := \prod_{\alpha \in \mathcal{P}} X_\alpha$ subgroup of U.
- A pattern $\mathcal{K}$ contained in a pattern $\mathcal{P}$ is *normal* in $\mathcal{P}$, or $\mathcal{K} \trianglelefteq \mathcal{P}$, if
$$\alpha \in \mathcal{K},\ \beta \in \mathcal{P},\ \alpha + \beta \in \Phi^+ \Longrightarrow \alpha + \beta \in \mathcal{K}.$$
- Fact 2: $\mathcal{K} \subseteq \mathcal{P}$ patterns. Then $\mathcal{K} \trianglelefteq \mathcal{P} \Longleftrightarrow X_{\mathcal{K}} \trianglelefteq X_{\mathcal{P}}$.

- A *quattern* $\mathcal{S} \subseteq \Phi^+$ is a set $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, with $\mathcal{P}$ pattern and $\mathcal{K} \trianglelefteq \mathcal{P}$.
- We define the *quattern group* associated to $\mathcal{S}$,
$$X_{\mathcal{S}} := X_{\mathcal{P}} / X_{\mathcal{K}}.$$

For a quattern $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, we define:

- the *central root support* in $\mathcal{S}$ [central (nontrivial) root subgroups in $X_{\mathcal{S}}$],
$$\mathcal{Z}(\mathcal{S}) = \{\gamma \in \mathcal{S} \mid 1 \neq X_\gamma \subseteq Z(X_{\mathcal{S}})\},$$
- the set of irreducible characters of $X_{\mathcal{S}}$ *supported on* $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$,
$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} := \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\gamma \not\subseteq \ker(\chi) \text{ for every } \gamma \in \mathcal{Z}\}.$$

We are going to give a partition of $\mathrm{Irr}(U)$ in terms of quatterns.

- Take an *antichain* $\Sigma \subseteq \Phi^+$, that is,

$$\gamma_1, \gamma_2 \in \Sigma \implies \gamma_1 \not\preceq \gamma_2 \text{ and } \gamma_2 \not\preceq \gamma_1.$$

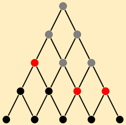
- Define the *standard quattern* $\mathcal{S}_\Sigma := \Phi^+ \setminus \mathcal{K}_\Sigma$, where

$$\mathcal{K}_\Sigma := \{\delta \in \Phi^+ \mid \delta \not\preceq \gamma \text{ for every } \gamma \in \Sigma\}.$$

- We define the following subset of $\mathrm{Irr}(U)$,

$$\mathrm{Irr}(U)_\Sigma := \mathrm{Inf}_{X_{\mathcal{S}_\Sigma}}^{\Phi^+}\left(\mathrm{Irr}(X_{\mathcal{S}_\Sigma})_\Sigma\right).$$

Ex. $U = \mathrm{UA}_5(q)$

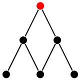


$\Sigma = \{\alpha_8, \alpha_9, \alpha_{10}\}$
$\mathcal{K}_\Sigma = \{\alpha_{11}, \dots, \alpha_{15}\}$
$\mathcal{S}_\Sigma = \{\alpha_1, \dots, \alpha_{10}\}$
$\mathcal{Z}(\mathcal{S}_\Sigma) = \Sigma$

Proposition (Himstedt, Le, Magaard '15)

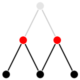
$$\mathrm{Irr}(U) = \bigsqcup_{\Sigma \text{ antichain in } \Phi^+} \mathrm{Irr}(U)_\Sigma.$$

For instance, to study $\mathrm{Irr}(\mathrm{UA}_3(q))$, enough to study



$\mathrm{Irr}(\mathrm{UA}_3(q))_\Sigma$, $\Sigma = \{\alpha_6\}$

and



$\mathrm{Irr}(\mathrm{UA}_3(q))_\Sigma$, $\Sigma = \{\alpha_4, \alpha_5\}$

- Recall $\mathrm{Irr}(\mathrm{UA}_2(q))$ in terms of quatterns. Put $\alpha = \alpha_1$, $\beta = \alpha_2$ and $\gamma = \alpha_3$.

  (i) $\gamma \in \mathcal{Z}(\Phi^+)$, (ii) $\alpha + \beta = \gamma$, $[(iii)$ $\delta + \beta \notin \Phi^+$ for all $\delta \in \Phi^+ \setminus \{\alpha\}]$.

  Then $\mathrm{Irr}(\mathrm{UA}_2(q))_{\{\gamma\}} = \{\mathrm{Ind}^\alpha \, \mathrm{Inf}_\beta \, \lambda \mid \lambda \in \mathrm{Irr}(X_\gamma) \setminus \{1_{X_\gamma}\}\}$.

Proposition

Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be a quattern, and $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$. Assume $\alpha, \beta, \gamma \in \mathcal{S}$ exist with

(i) $\gamma \in \mathcal{Z}$;

(ii) $\alpha + \beta = \gamma$; and

(iii) β is *well-behaved*, that is $\delta + \beta \notin \mathcal{S}$ for every $\delta \in \mathcal{S} \setminus \{\alpha\}$.

Then if $\mathcal{S}' := \mathcal{P}' \setminus \mathcal{K}'$, with $\mathcal{P}' = \mathcal{P} \setminus \{\alpha\}$ and $\mathcal{K}' = \mathcal{K} \cup \{\beta\}$, we have a bijection

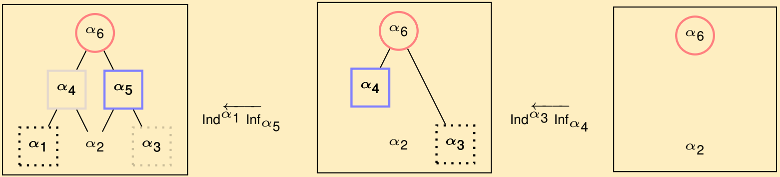
$$\mathrm{Ind}^\alpha \, \mathrm{Inf}_\beta : \mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}} \longrightarrow \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}.$$

- Call $\mathcal{S}$ *core* if it cannot be reduced further by applying the above method.

If $\mathcal{S}$ is an abelian core (i.e. $X_{\mathcal{S}}$ abelian), then $\mathrm{Irr}(U)_\Sigma$ is fully parametrized!

- We apply the previous proposition to compute

$$\mathrm{Irr}(\mathrm{UA}_3(q))_\Sigma, \quad \text{where} \quad \Sigma = \{\alpha_6\}.$$



$\mathcal{S} = \{\alpha_1, \ldots, \alpha_6\}$
$\beta = \alpha_5$, $\alpha = \alpha_1$

$\mathcal{S}' = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$
$\beta = \alpha_4$, $\alpha = \alpha_3$

$\mathcal{S}'' = \{\alpha_2, \alpha_6\}$
abelian core

Let us define

- $\lambda_{b_2}^{a_6} \in \mathrm{Irr}(X_{\alpha_2} \times X_{\alpha_6})$, for $b_2 \in \mathbb{F}_q$ and $a_6 \in \mathbb{F}_q^\times$.
- $\chi_{b_2}^{a_6} = \mathrm{Ind}^{\alpha_1, \alpha_3} \mathrm{Inf}_{\alpha_4, \alpha_5} \lambda_{b_2}^{a_6}$.

Then

$$\boxed{\mathrm{Irr}(\mathrm{UA}_3(q))_{\{\alpha_6\}} = \{\chi_{b_2}^{a_6} \mid b_2 \in \mathbb{F}_q, a_6 \in \mathbb{F}_q^\times\}}$$

consists of $q(q-1)$ irreducible characters of degree q^2.

- With the use of CHEVIE, we manage to parametrize $\mathrm{Irr}(U(q))$ for every group of rank 6 or less, *up to nonabelian cores*. **How many of them?**

Y_r, $r \leq 3$
0

B_4	C_4	D_4	F_4
1	0	1	6

B_5	C_5	D_5
10	1	7

B_6	C_6	D_6	E_6
95	22	55	156

- **Question.** How to study a nonabelian core?

Example: $\mathrm{UD}_4(q)$, $\Sigma = \{\alpha_8, \alpha_9, \alpha_{10}\}$

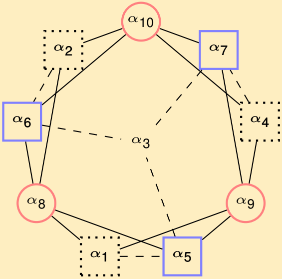
- Let $Z = X_{\alpha_8} \times X_{\alpha_9} \times X_{\alpha_{10}}$. Take $\lambda \in \mathrm{Irr}(Z)_{\{\Sigma\}}$.
- Inflate λ to $\hat{\lambda}$ over YZ, where $Y = X_{\alpha_5} X_{\alpha_6} X_{\alpha_7}$.
- $X = X_{\alpha_1} X_{\alpha_2} X_{\alpha_4}$ transversal of YZ in $X_{\mathcal{S} \setminus \{\alpha_3\}}$. Then

$$I_X(\hat{\lambda}) = 1 \text{ if } p \geq 3, \qquad |I_X(\hat{\lambda})| = q \text{ if } p = 2.$$

- Action of X_{α_3}: get $g(x) \in \mathbb{Z}_{\geq 0}[x]$ with

$$k(\mathrm{UD}_4(q)) = \begin{cases} g(q-1) & \text{if } p \geq 3, \\ g(q-1) + 3(q-1)^4 & \text{if } p = 2 \end{cases}$$

If $p = 2$, then $\chi(1) = q^3/2$ for a $\chi \in \mathrm{Irr}(\mathrm{UD}_4(q))$.



Theorem (Goodwin, Le, Magaard, P '16)

Let $U(q)$ be a Sylow p-subgroup of a group of Lie type of rank 4 or less, and p a not very bad prime for $G(q)$. The irreducible characters of $U(q)$ are completely parametrized.
Moreover, each irreducible character χ of $U(q)$ is parametrized as an inflation, followed by an induction of a character of a quattern of $U(q)$.

In particular, if $U = \mathrm{UF}_4(q)$, then

- We find an explicit polynomial $g(x) \in \mathbb{Z}_{\geq 0}[x]$, such that

$$k(\mathrm{UF}_4(q)) = \begin{cases} g(q-1) & \text{if } p \geq 5, \\ g(q-1) + 2(q-1)^4(q+3) & \text{if } p = 3; \end{cases}$$

- for $p = 3$, we find $\chi \in \mathrm{Irr}(\mathrm{UF}_4(q))$ such that $\chi(1) = q^4/3$;
- moreover, if $p = 3$ then

$$k(\mathrm{UF}_4(q), q^4), k(\mathrm{UF}_4(q), q^4/3) \in \mathbb{Z}_{\geq 0}[\tfrac{1}{2}(q-1)] \setminus \mathbb{Z}_{\geq 0}[q-1].$$

More nonabelian cores

Other examples of nonabelian cores are given in [Le, Magaard '15]. They arise from the following antichains.

- $\Sigma = \{\alpha_{17}, \ldots, \alpha_{21}\}$ in $\mathrm{UE}_6(q)$. Yields degrees $q^7/3$ if $p = 3$.
- $\Sigma = \{\alpha_{37}, \ldots, \alpha_{43}\}$ in $\mathrm{UE}_8(q)$. Yields degrees $q^{16}/5$ if $p = 5$.

Recall that we find characters

- of degree $q/2$ in $\mathrm{UB}_r(2^f)$ and $\mathrm{UC}_r(2^f)$.
- of degree $q/2$ in $\mathrm{UG}_2(2^f)$, and $q/3$ in $\mathrm{UG}_2(3^f)$. [Himstedt, Huang '09]

- Looking at subsystems of root systems, we can then formulate the following statement.

  Let p be a bad prime for a split G(q). Then there exist $\chi \in \mathrm{Irr}(U(q))$ and $d \in \mathbb{Z}_{\geq 1}$, such that for every q power of p we have $\chi(1) = q^d/p$.

- Moreover, if $G(q)$ is not of type E, and p is at least the Coxeter number of G, then every character degree is of the form q^d for some $d \in \mathbb{Z}_{\geq 0}$ [Goodwin, Mosch, Röhrle '16].

Directions for future work

- **Isomorphism of nonabelian cores**. In rank 6 we have hundreds of nonabelian cores. We can define isomorphism of quatterns, and reduce the number of classes of irreducible characters to parametrize. $\mathrm{Irr}(\mathrm{UD}_6(q))$ and $\mathrm{Irr}(\mathrm{UE}_6(q))$: work in progress.
- **Adapt to very bad primes**. By looking at Chevalley relations in $U(q)$ and at root lengths in Φ^+, get an argument to remove the assumption of p not very bad [Faltings; Le]. This would lead to a parametrization of $\mathrm{Irr}(\mathrm{UB}_r(2^f))$ and $\mathrm{Irr}(\mathrm{UC}_r(2^f))$ in small rank.
- **Twisted groups**. The construction of root subgroups is more complicated in this case. The set $\mathrm{Irr}(\mathrm{U}\,{}^3\mathrm{D}_4(q^3))$ is fully parametrized in [Le '13]. Aim: develop algorithmic approach in CHEVIE, for instance to study $\mathrm{Irr}(\mathrm{U}\,{}^2\mathrm{E}_6(q^2))$.
- **Towards** $\mathrm{UE}_8(q)$. The precise number of nonabelian cores is not known in this case ($\sim 10^7$). Investigation would require more sophisticated computational and character-theoretical methods.