The block graph of a finite group

Alessandro Paolini

joint with J. Brough and Y. Liu

Technische Universität Kaiserslautern

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G finite group. $p > 0$ prime, $p \mid |G|$.

Problem [Brauer '79]

Given a prime number p, find the relations between the properties of the *p-blocks* of characters of a finite group G and *structural* properties of G.

- Let R be the ring of algebraic integers in $\mathbb{C}$. If $\chi \in \mathrm{Irr}(G)$, then

$$\omega_\chi(x) := \frac{\chi(x)}{\chi(1)}|x^G| \in R \quad \text{for all } x \in G.$$

- Let $M \subseteq R$ with $pR \subseteq M$. Then $R/M \cong k$ with char $k = p$. We obtain the corresponding decomposition into *p-blocks*,

$$kG = B_1 \oplus \cdots \oplus B_m.$$

- This yields a partition of $\mathrm{Irr}(G)$ for each p. Let $\chi, \psi \in \mathrm{Irr}(G)$.

$$\mathrm{Irr}(G) = \bigsqcup_{B \text{ } p\text{-block of } G} \mathrm{Irr}(B), \qquad \chi \overset{\text{same}}{\underset{p\text{-block}}{\sim}} \psi \iff \omega_\chi \equiv \omega_\psi \mod M.$$

- Finally, we define the *principal p-block* $B_0(G)_p$ of G to be the unique p-block B of G such that $\mathrm{Irr}(B)$ contains the trivial character.

- If $p \nmid |G|$ or char $k = 0$, get much structural global info from the character table, for instance:
  - obtain $Z(G)$, namely $\{g \in G \mid |\chi(g)| = \chi(1) \text{ for every } \chi \in \mathrm{Irr}(G)\}$
  - get every $N \trianglelefteq G$ as $\cap_{\chi \in \mathcal{I}} \ker(\chi)$, for each $\mathcal{I} \subseteq \mathrm{Irr}(G)$, where $\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$
- We look at p-blocks when $p \mid |G|$ for further local information .

Proposition

Assume that there exists a p-block B such that $B = \{\chi\}$. *Then* $O_p(G) = 1$.

Theorem (Harris '85)

$B_0(G)_p$ *is the only p-block if and only if*

- $O_p(G) = F^*(G)$ *if* $p \geq 3$; *(*$F^*(G)$ *is the generalized Fitting subgroup of G)*
- $O_{2'}(G) = 1$ *and all components of G are of type* M_{22} *and* M_{24}, *if* $p = 2$.

- Compare now blocks at different primes p and r.

Theorem (Navarro, Willems '97)

$p, r \mid |G|$. Assume that there exist a p-block B_1 and an r-block B_2 such that $\mathrm{Irr}(B_1) = \mathrm{Irr}(B_2)$. If G is p-solvable or r-solvable, then $\mathrm{Irr}(B_1) = \{\chi\} = \mathrm{Irr}(B_2)$.

p- or r-solvability is necessary (Bessenrodt)

$G = 6.A_7$. There exist a 5-block B_1 and a 7-block B_2, with $\mathrm{Irr}(B_1) = \mathrm{Irr}(B_2)$, and $|\mathrm{Irr}(B_1)| = 5$. The blocks B_1 and B_2 are *not* principal blocks.

Our focus now is on the study of principal blocks $B_0(G)_p$, for $p \mid |G|$.

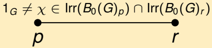
Theorem (Bessenrodt, Navarro, Olsson, Tiep '07)

Let p, r such that $\mathrm{Irr}(B_0(G)_p) = \mathrm{Irr}(B_0(G)_r)$. Then neither p nor r divide $|G|$. In particular, $\mathrm{Irr}(B_0(G)_p)$ just contains the trivial character of G.

- Criteria for nilpotency, solvability, simplicity in terms of principal blocks related to different primes motivate the definition of *block graphs*.

Main definition: block graph $\Gamma_B(G)$

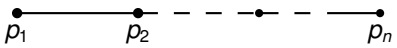
- vertices: $\pi(G) = \{p \text{ prime } \mid p \text{ divides } |G|\}$;
- $p \neq r$ are linked iff there exists a character χ in $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_r)$ with $\chi \neq 1_G$.



Theorem (Block graphs of nilpotent groups, Bessenrodt-Zhang ’08)

Let $|G| = p_1^{a_1} \cdots p_n^{a_n}$. Then G is nilpotent if and only if $\mathrm{Irr}(B_0(G)_{p_i}) \cap \mathrm{Irr}(B_0(G)_{p_j}) = 1$ for every $i \neq j$.

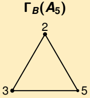
- *Remark:* no universal bound on *diam*$(\Gamma_B(G))$. Take $G = M_1 \times \cdots \times M_{n-1}$, with each of the M_i non-nilpotent, $|M_i| = p_i p_{i+1}$.

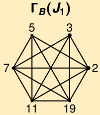


Theorem ($\Gamma_B(S)$ for S alternating or sporadic, Bessenrodt-Zhang ’08)

(i) *If $n \geq 4$, then $\Gamma_B(Alt(n))$ is a complete graph.*

(ii) *S sporadic. Then $\Gamma_B(S)$ is complete iff $S \neq J_1$ $[(p,r) = (3,5)]$ and $S \neq J_4$, $[(p,r) = (5,7)]$.*





- **Question:** can we complete the determination of block graphs of finite simple groups? YES!

Theorem (Brough, Liu, P. '17)

Let S be a finite simple group of Lie type. Then $\Gamma_B(S)$ is complete.

Idea of proof. Fixed $\ell_1, \ell_2 \mid |G|$, need to find $1 \neq \chi \in \mathrm{Irr}(B_0(S))_{\ell_1} \cap \mathrm{Irr}(B_0(S))_{\ell_2}$.

- Let $\mathbb{G}$ be the algebraic group with Frobenius endomorphism F over $\mathbb{F}_q$ associated to S. If F is very twisted, the claim follows from [Hiss '10].
- Let F be not very twisted. Define

$$e_i := \text{multiplicative order of } q \begin{cases} \text{modulo } \ell_i \text{ if } \ell_i \text{ is odd,} \\ \text{modulo } 4 \text{ if } \ell_i = 2. \end{cases}$$

- We construct Levi subgroups $\mathbb{L}_i$ of $\mathbb{G}$, $i = 1, 2$, centralizers of *Sylow e_i-tori*. Let $R_{\mathbb{L}_i}^{\mathbb{G}}(1_{\mathbb{L}_i^F})$ be the induced (virtual) *Lusztig character*. We use:

Theorem (Kessar, Malle '15)

The constituents of $R_{\mathbb{L}_i}^{\mathbb{G}}(1_{\mathbb{L}_i^F})$ lie inside $B_0(G)_{\ell_i}$, $i = 1, 2$.

- The claim follows from $\langle R_{\mathbb{L}_1}^{\mathbb{G}}(1_{\mathbb{L}_1^F}), R_{\mathbb{L}_2}^{\mathbb{G}}(1_{\mathbb{L}_2^F})\rangle = 0$ and $\langle R_{\mathbb{L}_i}^{\mathbb{G}}(1_{\mathbb{L}_i^F}), 1_{\mathbb{G}^F}\rangle = 1$.

- We would like to detect the solvability of G via its block graph.

Theorem (Brough, Liu, P. '17)

Let G be a finite group, $p \mid |G|$. If $\Gamma_B(G)$ does not contain triangles with vertex p, then G is p-solvable. In particular (by the Feit–Thompson theorem) if $\Gamma_B(G)$ does not contain triangles with vertex 2, then G is solvable.

The viceversa does not hold. $G = C_5^3 \rtimes \mathbf{Sym}(3)$

Bessenrodt–Zhang '11: if $\pi(G) = \pi_1 \sqcup \pi_2$, then π_1 and π_2 are disconnected in $\Gamma_B(G)$ if and only if $G = O_{\pi_1}(G) \times O_{\pi_2}(G)$.

Idea of proof. Reduction argument.

- If G is minimal not p-solvable, there is a minimal normal subgroup of G isomorphic to S^t with S nonabelian simple, $p \mid |S|$.
- Let $A := \mathrm{Aut}(S)$ and $M := A^t \cap G$. Then $C_M(S) \trianglelefteq N_M(S) = M$, and $\overline{M} := M/C_M(S)$ is almost simple with $\mathrm{Soc}(\overline{M}) = S$, that is, $S \leq \overline{M} \leq A$.
- S not of Lie type: [Bessenrodt-Zhang '08]. S of Lie type: via *Zsigmondy primes*, find $r, \ell \in \pi(S)$ such that $\{p, r, \ell\}$ is a triangle in $\Gamma_B(G)$.